Optimal Harvesting of a Replenishable Resource in a Model with Logistic Growth

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In 1931 Harold Hotelling wrote that the world might one day have to survive without those commodities which are produced with the aid of exhaustible resources. Since the late 1960's Cummings (1969), Anderson (1972), Vousden (1973), Smith (1974), Solow (1975), Lusky (1975) and Hu (1978) have suggested assorted recycling, pricing, taxation and Golden Age rules to alleviate the seemingly inevitable depletion of many types of resources. A related, and perhaps as serious, problem is that of replenishable resources, e.g., fish and timber, which has been addressed by Quirk and Smith (1969), Plourde (1970), Burt and Cummings (1970), Sampson (1976) and Nener (1976), among others.[1]

In Section I we specify a model which uniquely features logistic growth patterns for both the resource and human populations.[2] The model is then examined for its equilibrium and stability properties. We also present the derivation and interpretation of an op-

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1 See also the *Review of Economic Studies* Symposium (1974).

2 The model is similar to the preditor-prey models in the biological sciences. For example, and for early discussions of the logistic growth rule, see Verhulst (1838), Pearl, Raymond and Reed (1920), Lotka (1925) and Volterra (1931). Plourde (1971) takes a similar approach to that utilized in this paper but assumes a constant human population.

timal control rule. Following Vousden (1973), mathematical simplicity is enhanced by omitting capital from the production function, leaving the resource and labor as the only inputs; and a constant state of technology is assumed.

I. The Model

Consider a preditor-prey model with a replenishable resource R and people N where the rates of change in their two populations depends either partly or completely upon their sizes. For example, in any time interval the quantity of the resource will change due to the birth and death of individuals in the population and the destruction of the resource due to encounter with humans, who may destroy the resource either directly via consumption and/or production activities or indirectly via pollution, etc., where we assume that the rate of unnatural death of the resource depends upon the frequency of encounters between the two populations and is hence proportional to RN. Although the possibility is not treated in this paper, the rate of increase in the human population could be specified to be proportional, a la Malthus, to its frequency of encounters with the resource population.

The Pearl-Verhulst equations for the growth rates of the two populations are accordingly:

$$\dot{R} = \alpha R - \frac{\alpha}{b} R^2 - \beta RN \tag{1}$$

$$\dot{N} = \gamma N - \frac{\gamma}{a} N^2 \tag{2}$$

The equilibrium values of the two populations can be found by setting (1) and (2) equal to zero, i.e., N can take the values 0 or a.[3] We reject the first possibility as uninteresting so that $\overline{N} = a$ in equilibrium. In this case the resource population can also assume two values, namely

$$\overline{R} = 0, b\left(1 - \frac{a\beta}{\alpha}\right) \tag{3}$$

3 It is possible to assure that all equilibria are positive by adding arbitrarily small constant terms to (1) and (2).

Checking next for stability, total differentiation of (1) and (2) yields

$$\begin{pmatrix} d\dot{R} \\ \\ d\dot{N} \end{pmatrix} = \begin{pmatrix} \alpha - \frac{2\alpha}{b}R - \beta N - \beta R & \\ \\ 0 & \gamma - \frac{2\gamma}{a}N \end{pmatrix} \begin{pmatrix} dR \\ \\ dN \end{pmatrix} \tag{4}$$

According to the Routh-Hurwitz conditions, a necessary and sufficient condition for local asymptotic stability is that both terms on the main diagonal be negative. In addition, Olech's (1963) theorem asserts that equilibrium will be globally stable if their product is not equal to zero everywhere. Substituting into (4) for the equilibrium values of R and N with the resource totally depleted provides the Jacobian matrix

$$J = \begin{pmatrix} \alpha - a\beta & 0 \\ \\ 0 & -\gamma \end{pmatrix} \tag{5}$$

which may or may not be asymtotically stable. On the other hand, the potentially positive equilibrium solution for the resource population results in

$$J = \begin{pmatrix} -(\alpha - a\beta) & -b\beta\left(1 - \frac{a\beta}{\alpha}\right) \\ \\ 0 & -\gamma \end{pmatrix} \tag{6}$$

Noting the sign change from (5), we conclude that at least one of the equilibria is stable. If the upper righthand term in (6) is negative, the trajectory moves in a counter-clockwise spiral when the upper lefthand term is negative. If the term is positive, the trajectory is a clockwise spiral. In either case, the solution is seen to be periodic.

II. Optimal Control

It will be arithmetically helpful to eliminate one of the state variables. Accordingly, we select N and note that the righthand side of (2) equals zero when N = a, which is where growth ceases on the logistics curve. When all the terms of (2) involving N are collected and decomposed into partial fractions, we have

$$\left(\frac{1}{N}+\frac{1}{a-N}\right) dN - \gamma dt \tag{7}$$

or

$$\ln\left(\frac{N}{a-N}\right) = \gamma t + c \tag{8}$$

The solution is

$$N = \frac{a}{1+e^{-\gamma t}} \tag{9}$$

where $\lim_{t \to \infty} N = a$. Substituting (9) into (1) results in

$$\dot{R} = \left(\alpha - \frac{a\beta}{1+e^{-\gamma t}}\right) R - \frac{\alpha}{b} R^2 \tag{10}$$

We assume that the economic planner wishes to maximize the discounted value of a utility function whose sole argument is consumption.[4] Moreover, the consumer good or state variable is the end product of a production function incorporating labor and the amount of the resource diverted to human use, i.e.,

4 In the case of some resources, e.g., wilderness, it might be desirable to include the resource stock in a Sierra Club utility function.

$$C = F(N, \tilde{R})$$
$$= F(N, \beta NR)$$
$$= F(N, \beta R \frac{a}{1+e^{-\gamma t}}) \qquad (11)$$

where F is assumed to be well behaved with the usual signs and appropriate limits on the derivatives.[5] It follows that the present value utility function is

$$U = U[F(N, \beta R \frac{a}{1+e^{-\gamma t}})] \qquad (12)$$

which is also assumed to be strictly concave. Hence the planner's problem is to select a rule for the policy instrument β which maximizes the objective functional

$$\text{Max } J = {}_0\int^{\infty} e^{-\delta t}\, U\,[F(N, \beta R \frac{a}{1+e^{-\gamma t}})]\; dt \qquad (13)$$

subject to

$$\dot{R} = (\alpha - \frac{\beta a}{1+e^{-\gamma t}})\, R - \frac{\alpha}{b} R^2 \qquad (14)$$

$$R(0) = R_0 \qquad (15)$$

β is piecewise continuous

where δ is an appropriate (however determined) discount rate; where ∞ is the termination point of the control; and where $R(0) = R_0$ is the initial condition.[6] The present-value Hamiltonian is

5 $\tilde{R} = \beta NR$ is the amount of resource diverted to human use. This is the third term on the right hand side of equation (1).

6 The control on β may be direct, e.g., quotas or restrictions on harvesting techniques, or indirect via taxation or pricing policies.

$$H = U[F(N, \beta R \frac{a}{1+e^{-\gamma t}})] + \lambda [(\alpha - \frac{a\beta}{1+e^{-\gamma t}}) R - \frac{\alpha}{b} R^2] \tag{16}$$

where δ is a constate or auxiliary variable. Differentiating H with respect to the control and simplifying provides

$$\lambda = U_c F_r \tag{17}$$

which states that the shadow price or marginal cost of using up the resource in the productive process should be equal to the marginal utility derived from its use in production.[7] The critical control β is interior to the set of admissable β's since the utility and production functions are both strictly concave by assumption.

Finally, we need to eliminate the nonobservable auxiliary variable λ with the aid of the canonical equation

$$\dot{\lambda} = - \frac{\partial H}{\partial R} + \lambda \delta$$

$$= - U_c F_r \beta N + \lambda (\delta - \alpha + \beta N + \frac{2\alpha}{b} R) \tag{18}$$

Substituting from the control rule, we have

7 It might be desirable to include technical change in the model. For example, suppose that resource-saving technology occurs at the exogenous rate θ so that the production function becomes

$$C = F(N, \frac{a\beta R}{e^{-\theta t}(1+e^{\gamma t})})$$

The control rule is now

$$\lambda e^{\theta t} = U_c F_r$$

which states that utilization of the resource in production of the consumption good must increase at a more rapid rate than would otherwise be the case. A more realistic approach might be to introduce Kennedy-induced technical change as the costs of resource harvesting increase over time.

$$\frac{\dot{\lambda}}{\lambda} = \delta - \alpha + \frac{2\alpha}{b} R \tag{19}$$

Totally differentiating the control rule yields

$$\dot{\lambda} = F_r U_{cc} \dot{C} + U_c F_{rr} \dot{R} \tag{20}$$

and

$$\frac{\dot{\lambda}}{\lambda} = \frac{U_{cc}}{U_c} \dot{C} + \frac{F_{rr}}{F_r} \dot{R} \tag{21}$$

Setting the righthand sides of (19) and (21) equal, substituting for $\dot{R}$, and rearranging, provides

$$\dot{C} = \frac{U_c}{U_{cc}} [\delta - \alpha + \frac{2\alpha}{b} R - \frac{F_{rr}}{F_r} R (\alpha - \frac{\alpha\beta}{1 + e^{-\gamma t}} - \frac{\alpha}{b} R)] \tag{22}$$

Noting that

$$\sigma (R) = - \frac{F_{rr}}{F_r} R \tag{23}$$

and

$$\sigma (C) = - \frac{U_{cc}}{U_c} C \tag{24}$$

are the elasticities of the marginal productivity of the resource and the marginal utility of consumption, both of which are unambiguously positive, and with the optimal control we have the pair of differential equations

$$\dot{R} = (\alpha - \frac{a\beta}{1 + e^{-\gamma t}}) R - \frac{\alpha}{b} R^2 \tag{25}$$

$$\dot{C} = -\frac{C}{\sigma(C)} [\delta - \alpha + \frac{2\alpha}{b} R + \sigma(R) (\alpha - \frac{a\beta}{1 + e^{-\gamma t}} - \frac{\alpha}{b} R)] \tag{26}$$

From the standpoint of stability and the steady-state properties of the model, it is necessary to assume that the human population has reached the top of its logistics curve and is hence constant. Moreover, arithmetic simplicity will be greatly enhanced by assuming that $\sigma(C) = \sigma(R) = 1$ so that
(26) becomes

$$\dot{C} = -C (\delta - a\beta + \frac{\alpha}{b} R) \tag{27}$$

Noting that the control instrument depends upon the product of U_c and F_r, the dynamic structure can be written

$$\dot{R} = \alpha R - a\beta (U_c F_r) R + \frac{\alpha}{b} R^2 \tag{28}$$

$$\dot{C} = -C [\delta - a\beta (U_c F_r) + \frac{\alpha}{b} R] \tag{29}$$

which, after differentiating, yields

$$\begin{pmatrix} d\dot{R} \\ \\ d\dot{C} \end{pmatrix} \begin{pmatrix} \alpha - a\beta' U_c F_{rr} R - a\beta - 2\frac{\alpha}{b} R, \\ - a\beta' F_r U_{cc} R \\ \\ aC\beta' U_c F_{rr} - \frac{\alpha}{b} C, \\ - (a\beta - \delta) + a\beta' F_r U_{cc} C - \frac{\alpha}{b} R \end{pmatrix} \begin{pmatrix} dR \\ \\ dC \end{pmatrix} \tag{30}$$

where $\beta'(U_c F_r) < 0$.

The phase diagram for (30) is depicted in Figure 1. The phase line for R demonstrates that a large stock of the resource can be maintained only at the cost of maintaining a low level of consumption whereas the phase line for consumption illustrates that equilbrium consumption will be low if the resource is in small supply and that a large stock of the resource can be maintained only at the expense of low consumption. The maximum sustainable level of consumption is at C', which is greater than that obtained from the optimal at C*. The arrows indicate the directions of change for the two variables in each of the six subregions. The solution at (R*, C*) is easily seen to be a saddle-point with the optimal trajectory depicted by the arrowed line. If the rule is set so that the phase lines intersect at (R', C'), the solution is not a saddle-point and is hence sub-optimal.[8]

The long-run equilibrium stock of the resource is obtained by setting $\dot{R}$ equal to zero in (28) in order to obtain

$$R^* = \frac{b}{\alpha}(\alpha - a\beta) \tag{31}$$

If the steady-state stock of the resource is to remain positive, it is required that $\alpha > a\beta$. Otherwise, the resource, replenishable or not, will be depleted or maintained at a zero supply. Moreover, setting C = O and sub-stituting R* into (29) yields the additional requirement that $\sigma + \alpha > 2a\beta$ if R* is to be positive in a steady state. The long-run equilibrium level of consumption is then easily found from the production function, given N (= a) and R*. The equilibrium can be best described as a modified Golden Rule in the sense that the resource-labor ratio, the output-resource ratio and hence the output-labor ratio all remain constant.

8 As shown by Plourde (1970), R', C' will be optimal if the discount rate is zero. Also, Stiglitz (1976) has advanced the intriging notion that monopolies extract resources at less-than-optimal rates. However, this position has been challenged by Tullock (1979) and Lewis, Matthews, and Burness (1978), who utilize counter-examples to show that monopolists may extract resources too rapidly under certain cost and demand conditions.

III. Conclusions

This paper has applied the maximum principle to a model in which both human and resource populations are assumed to grow according to the logistic rule, although it is a simple matter to alter the model so that the case of exhaustible resources can be handled. An optimal control rule was derived and its properties were examined, as well as those of the steady-state solution. Several possible extentions can be suggested. First, the real world is not deterministic so that it would be desirable to analyze the stochastic analogue of the model and perhaps utilize adaptive or stochastic programming to find an optimal control. A second interesting variant would be the situation in which two populations compete for a third resource, e.g., people and animal or fish life vying for space or land resources.

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