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# Reputation and Imperfect Information in the Debt Market*

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This paper investigates a dynamic incentive of a long-lived borrower in a debt market with adverse selection and moral hazard. In the short run, due to the lack of commitment by the borrower, either in the capital market breaks down, or at best socially desirable project is sacrificed against the less desirable project. We show the 'reputation effect' can act as means for the precommitment to the socially desirable project in repeated play. We present both finite and infinite horizon models. In finite horizon model, we show that if the discount factor is close to 1, and the time horizon is sufficiently long, the borrower chooses the socially desirable project in all but possibly the last few periods. In infinite horizon model, we show that as discount factor goes to 1, the reputation effect works in terms of payoffs.

## I. Introduction

This paper investigates a dynamic incentive problem of a long-lived borrower in a debt market with adverse selection and moral harzard.

The long-lived borrower is faced with an intertemporal decision problem of choosing between risky or safe project. The safe project guarantees some positive level of output with probability 1. The risky project may fail with some positive probability so that it returns nothing. If it succeeds, it gives a larger output than the safe project does. We assume that the safe project generates a larger expected surplus than the risky project, thereby it is a socially desirable project. In short-run, since the borrower can default, the risky project gives the higher expected payoff to the borrower than does the safe project for all

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possible interest rates. Therefore once the borrower gets a loan, he invests in the risky project. However, because the probability for success of risky project is too low, either the capital market breaks down, or at best the risky project is chosen, although the safe project is socially desirable. So economic inefficiency arises.

As one way to restore the economic efficiency, we introduce incomplete information about the type of borrower. We allow two more types of borrower, each of which has available one risky or one safe project only. The lenders can observe neither the type of borrower nor the project choice. The lenders can observe payback-default history only.

We present both finite and infinite horizon models. In finite horizon model, we show that if the discount factor is close to 1 and the time horizon is sufficiently long, in all Nash equilibria the borrower chooses the safe project in all but possibly the last few periods. Therefore, economic efficiency is restored except some last few periods.

This result, however, does not hold in infinite horizon model. We present an example where the borrower chooses the risky project infinitely often, although the discount factor is close to 1. In infinite model, instead of strategies, we tackle the problem in terms of payoffs. We show that as discount factor converge to one, the set of Nash equilibrium payoffs for the borrower converge to the payoff where the borrower chooses the safe project and the lenders quote interest rate accordingly. Therefore reputation works in terms of payoff. This result implies that even though the borrower may choose the risky project infinitely often, the 'density' of the periods in which the borrower chooses the risky project should go to zero.

Compared with usual reputation model in the literature, say Kreps and Wilson [1992b] where the uninformed party can observe the action chosen by the informed party, in our model the uninformed lenders have coarser information because they can observe payback-default history only. But our result shows that even under this coarser information structure, reputation will work.

In the literature, Diamond [1989] analyzes the similar model to ours. His analysis differs from ours in two aspects. First, his analysis is restricted to a finite horizon model. Second, his analysis assumes a fixed value for the discount factor. Typically, his equilibrium takes the following form: In early periods, the borrower chooses the risky project. If he successfully repays his debt for several periods, he recognizes that he has acquired good reputation. Then reputation becomes a valuable asset. Therefore, in order to keep reputation, he switches to the safe project until near the end of the game. Near the end, he switches back to the risky project. Therefore, in Diamond's analysis, it takes some time before reputation works. On the contrary, our result shows that for the case of discount factors close to one, reputation works from the beginning. We also show that in infinite horizon model, the same result holds in terms of payoffs.

In recent two papers, Fudenberg and Levine [1989, 1992] deal with reputation in infinitely repeated games where one long-lived player plays the game with a sequence of the short-lived players. For this class of games, they provide a lower bound for the set of Nash equilibrium payoffs for the long-lived player. Some of our results are the special cases of the general results in Fudenberg and Levine [1989, 1992]. However, due to the simple structure of the model, we have a stronger result: In addition to the lower bound for the set of Nash equilibrium payoffs, we can provide an upper bound which is equal to the lower bound.

The paper proceeds as follows: In section 2, we specify the economic environment. In section 3, since this paper is stimulated by Diamond [1989], we briefly compare the models and recapitulate his results. Section 4 deals with finite horizon model. Since we can apply a backward induction in finite horizon model, we begin with the one-period problem and characterize its equilibrium. Then, we examine how reputation works in a sufficiently long finite horizon model with high discount factors.

Section 5 presents an infinite horizon model. We provide an example where the borrower with a choice of projects can choose the risky project infinitely often in equilibrium. Then in subsection 5.1, we analyze the case where there is zero prior probability that the borrower has a choice. In this case, the equilibrium interest rate is determined independently of the project choice by the borrower who has a choice. In subsection 5.2, we consider the full model where each type of borrower has positive prior probability. We show that as the discount factor goes to one, the set of Nash equilibrium payoffs for the borrower with a choice converge to the payoff that would accrue when the borrower can commit to choosing the safe project.

## II. The Basic Model

We adopt a model similar to one developed in Diamond [1989].

Following Fudenberg and Levine [1989, 1992], we formulate the model as a game between a long-lived borrower and a sequence of short-lived lenders.

The economy exists at a sequence of dates, indexed by $t=1,2\ldots,T$, where $T=\infty$ is not precluded. At each date there are two goods, an "input to production" good, and a consumption good. The input good cannot be consumed, but can be converted by production into the consumption good. All endowments are in input good, which is converted (in each period) into the consumption good and is then immediately consumed. Both goods are highly perishable; neither can be stored from one period to the next.

The lenders receive a positive amount input good in each period, and they have available a constant returns to scale technology for converting this good to the consumption good. This technology returns r units of consumption good per unit of input.

Since our analysis focuses on the reputation of the borrower, a passive role is assigned to the lenders. Formally we assume that lenders live for one period only. They lend or store their endowment good, receive returns, consume those returns and then pass out of existence, and a new generation of lenders flows in. Therefore, lenders do not develop reputation. Informally, any assumption that causes lenders to act myopically will do. The lenders are assumed to have identical, risk-neutral preferences.

The borrower also has risk-neutral preferences and seeks to maximize expected value of the discounted sum of his consumption, $E\{\Sigma_{t=1}^{T}\delta^{t-1}c_t\}$, where expectation is taken with respect to the project choice by borrower described below, $c_t$ is the borrower's level of consumption at period t, and $\delta$ is a discount factor, $0<\delta<1$.

The borrower receives no endowment. Instead, he has access to indivisible investment projects every period. There are two types of projects, risky and safe. The borrower is one of three types, which determines which types of projects he has available. The three types of borrower are:

- G-borrower, who has available one safe project only in each period. He can invest 1 unit and receives $G(>r)$ at the end of the period.
- B-borrower, who has available one risky project only in each period. He can invest 1 unit, and with probability $\pi<1$, the project succeeds. In this case the return is $B(>G)$. With probability

$1-\pi$, the project fails and the return is zero.

- BG-borrower, who has one of each type of project available in each period, but who must choose between the projects.[1] BG-borrower cannot invest in both projects at any one time.

The type is private information to borrower. As specified above, G-borrower or B-borrower has no choice.[2] It is BG-borrower who has a real choice and who therefore has the incentive to develop a reputation.

Let P be the prior assessement of the lenders concerning the type of borrower. $P=(P_g, P_b, P_{bg})$, where $P_g$, $P_b$, and $P_{bg}$ are prior probabilities of G, B and BG-borrower, respectively. The prior P is common knowledge. Each of $P_g$, $P_b$ and $P_{bg}$ is strictly positive.

As mentioned earlier, the borrower can only invest one unit of the endowment good. We assume that the total endowment in any period of the input good exceeds the requirements of the borrower, and thus the lenders' constant returns to scale technology is used at the margin in each period.

The game played between the sequence of lenders and the borrower takes the following form. At each period, $t=1,2,\ldots$, the lenders offer a financial contract to the borrower.[3]

Then the borrower decides whether to accept or reject this contract. If the borrower gets a loan, he chooses (if he has a choice) his project. Finally the borrower observes the return of his investment and, if he can, pays back to the lenders, consuming any consumption good that remains from the returns to his investment. If the borrower cannot repay the loan, he defaults and consumes nothing.[4]

1 Diamond assumes that B-borrower and BG-borrower have different probabilities for success. For simplicity, we assume that they have the same probability.

2 Borrowing the terminology from Fudenberg and Levine (1992), G and B-borrower are commitment types.

3 Due to the informational asymmetry, an equity contract cannot be implemented. Thus we look at simple debt contracts, with enforceability provisions to be described. We work with a formulation where the uninformed lenders offer a contract which the borrower accepts or rejects. Alternatively, one could study a model where the informed borrower offers a contract.

Our formulation has two advantages over the second formulation. First, since the uninformed lenders make all the offers, we do not worry about the "out-of-equilibrium" beliefs. Second, we do not need an explicit tie-breaking rule which should apply when the lenders are indifferent between lending or not. Furthermore, since the lenders compete in Bertrand-like way, their equilibrium payoff should be exactly equal to r.

4 For simplicity, we assume that at the final stage of the game, the borrower honestly pays back if he can. Anternatively, we may think in terms of the loan covenant where the

At the beginning of period, the borrower knows his type, his choice of projects and the outcome of project up to period $t-1$, and the payment history up to period $t-1$. The lenders at period t cannot observe the borrower's type, his choice of projects up to perod $t-1$, or the return of the projects. The only information that lenders possess at period t is payback-default history up to period $t-1$, on the basis of which they decide whether to offer a financial contract or not.

Since the lenders cannot observe the returns of the projects, an equity contract that specifies a fixed proportion of the return cannot be implemented. So we restrict our attention to debt contract with fixed required payment to lenders.

Let $r_t$ denote the fixed payment demanded by the lenders in period t. Since the lenders compete in Bertrand-like fashion, and the lenders' constant returns to scale technology is used at the margin in any equilibrium, along the equilibrium path the lenders quote $r_t$ such that $r_t$ multiplied by the probability of payback (which can be calculated by using the probability assessement concerning the type of the borrower and the project choice by each type of borrower at period t) is exactly equal to r.

Finally we introduce some inequalities needed for reputation to work. Throughout this paper, we assume that

$$(1) \qquad \pi(B-r)>G-r>\max\{0,\ \pi(B-\frac{r}{\pi})\}.$$

These inequalities provide us with the following basic considerations. By the first inequality, for any $r'>r$, $\pi(B-r')>G-r'$. This implies that the risky project is in BG-borrower's short run interests. Since the lenders know this, they quote interest rate, $r/\pi$. If $r/\pi>B$, the capital market breaks down. If $r/\pi\leqslant B$, BG-borrower gets a loan at interest rate $r/\pi$, and invests in the risky project. One implication of the second inequality is that $G>\pi B$. In other words, in expected term, the safe project produces more social surplus than the risky project does. In short run, due to lack of commitment by the borrower, there is a social efficiency loss in the sense that either the capital market breaks down or, at best socially desirable project is sacrificed against the less desirable project. The second inequality introduces a long-term consideration such that in repeated play, the social efficiency can be restored. Read this

lenders can and will have the assets of the borrower seized, if the borrower defaults, where the act of seizure partly destorys those assets, with the lenders getting whatever remains. In this case, the borrower will pay back the loan whenever he is able to do so.

second inequality as follows: If BG-borrower could commit himself to taking the safe project, he would net $G-r$. If he is assumed to take the risky project, he nets $\max\{0, \pi(B-r/\pi)\}$. Hence the second inequality suggests that BG-borrower would wish to precommit to the safe project, if the alternative is a presumption that he will take the risky project.

Our analysis shows how reputation can act as a means for this precommitment in repeated play so that the social efficiency can be restored. Indeed, the second inequality in (1) is crucial in order to show that as discount factor $\delta$ goes to 1, reputation becomes the most important factor which BG-borrower takes into consideration when making an optimal intertemporal decision.

## III. A Brief Recapitulation of Diamond

Since this paper is motivated by Diamond [1989], it is worthwhile to compare the assumptions in two models and to recapitaulate Diamond's analysis.

Diamond [1989] assumes that the first inequality of (1) holds; i.e., $G-r<\pi(B-r)$. In addition, he also assumes that $r/\pi>B$. Therefore, the capital market breaks down if the borrower turns out to be of type B or BG. Hence both the conflict between the borrower's and the lenders' interests and efficiency loss arise. Instead of $r/\pi>B$, we make a weaker assumption.

$$(2) \qquad G-r>\max\{0, \pi(B-\frac{r}{\pi})\}.$$

Clearly, if $r/\pi>B$ then $\max\{0, \pi(B-r/\pi)\}=0$, therefore the inequality holds trivially. But we do not rule out the case where $r/\pi\leqslant B$. As assumed in Diamond [1989], if $r/\pi>B$, in finite horizon model, the capital market breaks down if the borrower reveals himself to be of either B or BG. We do not exclude the possibility that $r/\pi\leqslant B$. In this case, the lenders are to quote an interest rate at least $r/\pi$, and B and BG-borrower surely choose the risky period, and get $\pi(B-r/\pi)$. Therefore, there is no conflict of interests between the lenders and the borrower, but efficiency loss still arises. In either case of $r/\pi>B$ or $r/\pi\leqslant B$, $\max\{0, \pi(B-r/\pi)\}$ is what a borrower can get at one period after he is revealed to be of either type B or type BG.

Diamond [1989] characterizes three different types of equilibria in finite horizon model and shows that equilibrium will be unique for a given parameter values. Following are three possible types of equilibria.

1. Reputation effects are always absent, in the sense that BG-borrower selects risky projects in all periods.

2. Reputation effects on borrowing costs are so strong that BG-borrower selects safe projects for all but possibly some last several periods.

3. Reputation effects serve to provide incentives to select safe projects, but a period of reputation acquisition is required when BG-borrower selects risky projects. If he successfully pays back by choosing risky projects, the interest rate falls over time and eventually reputation becomes such a valuable asset that he switches to safe projects for the rest of time except possibly some last several periods.

Diamond [1989] discussed the condition for type 1 equilibrium to prevail. The condition is

$$\frac{G-r}{1-\delta}<\frac{B-r}{1-\delta\pi}.$$

Under this condition, no matter how long the horizon is or how low the interest rate is (as long as it is still greater than r), the optimal choice for BG-borrower will be risky projects. In this case, BG-borrower always chooses a risky project not because he is so myopic. He fully recognizes what reputation buys him. He chooses risky project because the short run gain from choosing risky projects exceeds the long run gain from reputation-building.

As seen earlier, however, as the discount factor $\delta$ goes to one, the inequality must be reversed as long as $G-r>0$ and $\pi<1$. Therefore, for $\delta$ sufficiently close to one, type 1 equilibria disappear. This is due to the fact that as the discount factor goes to 1, reputation becomes more valuable so that always choosing risky projects cannot be supported as an equilibrium behavior. Our objective in this paper, roughly put, is that with sufficiently long time horizon and for $\delta$ sufficiently close to 1, only type 2 equilibria remain: type 3 equilibria also disappear.

## IV. Finite Horizon Model

In this section, we consider a finite horizon model. A finite horizon model has an advantage that we can apply a backward induction from the last period. So we begin to analyze the choice of project in the last period.

*A. One-period Model*

This subsection is devoted to analyzing the borrower's otpimal decision problem in one-period model.

The choice of project by BG-borrower will affect the interest rate that the lenders will quote, and the interest rate also influences the optimal choice of project by BG-borrower. So the equilibrium choice of project and the equilibrium interest rate must be determined simultaneously. For this purpose, we define two levels of interest rates, $r_{min}$, and $r_{max}$.

$$r_{min}=\frac{r}{P_b\pi+(1-P_b)},\ r_{max}=\frac{r}{P_g+(P_b+P_{bg})\pi}$$

$r_{min}$ ($r_{max}$, respectively) is the interest rate that the lenders will quote if they believe that BG-borrower will choose the safe project, (risky project, respectively).

By (1), for all $r'>r$, BG-borrower chooses the risky project in one-period problem. Since $r_{min}$ is calculated by assuming that BG-borrower would choose the safe project, it cannot be an equilibrium interest rate. Since the lenders know that BG-borrower chooses the risky project at every possible interest rate, the equilibrium interest rate is at least $r_{max}$.

The equilibrium in one-period problem takes the following forms. If $r_{max}<G$, the equilibrium is such that the lenders quote $r_{max}$, and all the types of borrower accept and BG-borrower chooses the risky project.

If $r_{max}\geqslant G$, two cases are possible. If $B<r/\pi$, the capital market breaks down. If $B\geqslant r/\pi$, the equilibrium is such that the lenders quote $r/\pi$, and G-borrower reject the offer, B and BG-borrower accept the offer and BG-borrower chooses the risky project. Before we proceed, we summaize our analysis of one-period problem and put it as proposition 1.

**Proposition 1.** Under assumption (1), the equilibrium in one-period model takes the following form.

1. For all values of $P_b$, $P_{bg}$, $P_g$ for which $r_{max}<G$, the lenders quote the interest rate $r_{max}$, all the borrowers accept and BG-borrower chooses the risky project.
2. If $r_{max}>G$, two cases are possible.
   (a) If $B<r/\pi$, the capital market breaks down.
   (b) If $B\geqslant r/p$, the lenders quote $r/\pi$, only B and BG-borrower accept and both choose the risky project.

### *B. Continuation Game with $P_b + P_{bg} = 1$*

The analysis of continuation game with $P_b + P_{bg} = 1$ is an important step in the analysis of the whole game because if BG-borrower defaults by choosing risky project, it is a conclusive proof (to future lenders) that the borrower is not of type G; i.e., $P_b + P_{bg} = 1$.

We will use the following observation so frequently that we put it as a lemma. Roughly put, it says that in the repeated game, due to the existence of B-borrower, there are no "unexpected information sets." Its proof is straightforward.

**Lemma 1.** Suppose that lenders observe only whether the borrower repays or defaults, whenever a loan is made, and suppose that whenever any type of borrower takes the loan contract offered, B-borrower do so. (Both conditions hold in our game.) Then (as long as the prior probability of type B is positive), every (finite) sequence of payback-default has positive probability, since every sequence is consistent with B-borrower. And therefore, after any (finite) sequence of payback-default, there is always strictly positive posterior probability that the borrower is of type B.

Proposition 2 characterizes the payoff to BG-borrower in the continuation game when $P_b + P_{bg} = 1$.

**Proposition 2.** Suppose that, in a Nash equilibrium, at some information set the lenders believe that the borrower is of either type B or BG with probability one. Then in the equilibrium continuation from that information set, the continuation payoff per period for the borrower is $\max\{0, \pi(B - r/\pi)\}$.

Proof: We prove this by induction on the number of period to go. In one-period problem, it trivially holds by (1). Suppose that this is true if there $T-1$ or fewer periods to go. With T periods to go, whatever strategy BG-borrower does at period T, since B-borrower has positive probability, by lemma 1 every information set for the lenders at period $T-1$ is reachable with positive probability. Therefore, at every information set at period $T-1$, the lenders retain the belief that $P_b + P_{bg} = 1$. By the induction hypothesis, BG-borrower chooses the risky project from period $T-1$ on, and $\max\{0, \pi(B - r/\pi)\}$ is the continuation payoff per period. Then the choice problem at period T just boils down to the one-period problem with $P_b + P_{bg} = 1$. That is, nothing BG-borrower does in this period (in or out of equilibrium) affects the continuation game. Hence BG-borrower will choose the risky project at period T, obtaining $\max\{0, \pi(B - r/\pi)\}$ at period T. Q.E.D.

If a borrower defaults at some period t, it shows that he is either of type B or BG. Hence a simple corollary to proposition 2 is following.

**Corollary:** If a borrower defaults at some period t, then his continuation payoff is $\max\{0, \pi(B-r/\pi)\}\Sigma_{k=1}^{t-1}\delta^{k-1}$.

Now we state and prove our main theorem for finite horizon model.

**Theorem 1.** Under the assumption (1), for all the prior probabilities P for which $r_{max}<G$, there exists $T^*$ and $\delta^*$ such that if T, the number of periods to go, is greater than $T^*$ and $\delta \geqslant \delta^*$, in any Nash equilibrium BG-borrower chooses the safe project in all but possibly $T^*$ periods of the game.

Proof: See Appendix A.

The intuition behind theorem 1 is clear. As $\delta$ is close to 1, the future becomes more important, hence in choosing the project, BG-borrower takes into consideration not only today but also tomorrow. In particular, if there are many periods to go, he has to pay more attention to the future. Premature choice of risky project may result in default so that he loses much more than by choosing the safe-project, paying back, and continuing to borrow with more favorable interest rate.

## V. Infinite Horizon Model

In infinite horizon model, since it is convenient to normalize the discounted sum of payoff by multiplying by $1-\delta$, we assume that the borrower is to maximize $(1-\delta)E(\Sigma_{t=1}^{\infty}\delta^{t-1}c_t)$.

First, we provide an example where even for discount factor close to 1, BG-borrower chooses the risky project infinitely often in equilibrium. For this, assume $G=2$, $r=1$, $B=3$, and $\pi=1/4$. First, we consider the cse where $P_{bg}=1$. In this case, since $r/\pi=4>B=3$, regardless of discount factors, there is a trivial equilibrium where the capital market breaks down. We call this type 1 equilibrium. The payoff for BG-borrower is zero in this equilibrium. Now think about the following pair of strategies: in every period, the lenders quote $r=1$ with belief that BG-borrower surely chooses the safe period. If BG-borrower chooses the risky project so that he defaults, thereafter, type 1 equilibrium is in effect. BG-borrower chooses the safe project with probability 1 in every period. We can easily show that this pair of strategy indeed constitutes a Nash equilibrium for $\delta \geqslant 1/2$. In this equilibrium, the payoff for BG-borrower is $1(=G\text{-}r)$. Call this type 2 equilibrium.

Now consider the case where $P_{bg}+P_g=1$, say $P_{bg}=P_g=1/2$. Think about the following pair of strategies: For BG-borrower, At period $t=2^n$, $n=1,2,\ldots$, BG-borrower chooses the risky project with probability 1. At period $t\neq 2^n$, $n=1,2,\ldots$, BG-borrower chooses the safe project with probability 1.

For the lenders, if the borrower has not yet defaulted, at period $t=2^n$, quote $r_t=(4^n+4)/(4^n+1)$. Otherwise, the lenders quote $r=1$. If the borrower defaults in the first time at period t such that $t=2^n$ for some n, type 2 equilibrium is in effect. If he defaults at other periods, type 1 equilibrium is in effect.

For discount factors close to 1, this pair of strategy is a Nash equilibrium. The point of this construction is that at period $t=2^n$, BG-borrower chooses the risky project because although he defaults, the continuation game is not so severe to him. However, if BG-borrower defaults at other periods, the continuation game is the most severe one to him, which prevents him from choosing the risky project. By construction, in this equilibrium, BG-borrower chooses the risky project infinitely often. However, as $\delta$ goes to 1, the equilibrium payoff goes to $1(=G\text{-}r)$. This comes from the fact that $(1-\delta)\Sigma_{n=1}^{\infty}\delta^{2n}$ goes to 0 as $\delta$ goes to 1. So the discounted sum of payoffs at periods $t=2^n$ does not affect the total payoff as $\delta$ goes to 1.[5]

As this example shows, we cannot expect that in infinite horizon model, for discount factors close 1, BG-borrower chooses the risky project at most finitely many times. Instead of strategies we tackle the problem in terms of payoffs. We will prove that as $\delta$ goes to 1, the set of Nash equilibrium payoffs for BG-borrower converges to $G-r$. Therefore, reputation effect will work as a means for the precommitment to the safe project in infinite horizon model at least in terms of payoff.

In the following two subsections, we will prove our theorem. Subsection 5.1 deals with the case where the prior probability of type BG is zero. This has an advantage that the sequence of equilibrium interest rates is determined independently of the choice by BG-borrower because the probability of type BG is assumed to be zero. Subsection 5.2 deals with the full model where each type has strictly positive prior probability. In this case, we also take into consideration the interaction be-

[5] If $P_b$ 0, let BG-borrower choose the safe project for long but finite periods so that he always pays back. (If BG-borrower defects during these period, type 1 equilibrium is in effect when the borrower defaults.) After long period history of payback, the posterior probability of B-borrower becomes sufficiently small. Then, similar construction where $P_{bg}+P_g=1$ works except that the interest quoted by the lenders becomes a little higher due to the existence of B-borrower.

tween equilibrium configuration of interest rates and the intertemporal choice by BG-borrower.

## *A. Infinite Horizon Model with $P_{bg}=0$*

In this subsection, we analyze the case where the prior probability of BG-borrower is zero, or $P_g + p_b = 1$. Even if the lenders believe that the borrower is of type B or G, we can ask how BG-borrower would act. We assume that both B-borrower and G-borrower have strictly positive prior probabilities.

In this formulation, since the prior probability of BG-borrower is zero, the equilibrium project choice by BG-borrower does not influence the determination of equilibrium interest rates.

First of all, we examine how the equilibrium interest rate is determined. At period $t=1$, given the prior probabilities of each type, $P_g$, $P_b$, and $\pi$, the probability of success for risky project, the probability of pay back is $P_g + \pi P_b$ so that lenders will quote at least $r/(P_g + \pi P_b)$, if they are willing to lend. Hence if $r_1 = r/(P_g + \pi P_b) < G$, $r_1$ is the equilibrium interest rate at period $t=1$ and both types of borrower get the money at this interest rate. If $r_1 = r(P_g + \pi P_b) \geqslant G$, then just as in Proposition 1, two cases are possible. If $B < r/\pi$, the market breaks down. If $B \geqslant r/\pi$, the only equilibrium is such that the lenders quote the interest rate $r/\pi$, and only B-borrower borrows and invests in the risky project. Since the case where $r_1$ is greater than G is uninteresting, from now on we assume

$$(3) \qquad r_1 = \frac{r}{P_g + \pi P_b} < G.$$

Under this assumption, we can determine the equilibrium interest rate at period t along the path where up to $t-1$ periods, the borrower has successfully paid back all loans.

Note that at each period, B-borrower can pay back with probability $\pi$. Therefore for B-borrower, the probability for successfully paying back for $t-1$ periods is $\pi^{t-1}$. Since G-borrower can pay back every period, for G-borrower the probability for successfully paying back for $t-1$ periods is 1. Hence the equilibrium interest rate after $t-1$ period successful payback is:

$$(4) \qquad r_t = r\,\frac{\pi^{t-1}\, P_b + P_g}{\pi^t\, P_b + P_g}$$

Note that $\{r_t^*\}_{t=1}^{\infty}$ monotonically decreases to r as t goes to $\infty$. This occurs because as time passes, B-borrower eventually defaults so that after sufficiently many periods of successful payback lenders are pretty sure that the borrower is of type G, therefore the interest rate is very close to r.

The decision problem facing BG-borrower is now reduced to a stationary discounted dynamic programming problem with the following transition probability structure for interest rate.[6]

$$P(r_{t+1}^* | r_t^*, \text{ safe project}) = 1.$$

$$P(r_{t+1}^* | r_t^*, \text{ risky project}) = \pi, \text{ and } P(r/\pi | r_t^*, \text{ risky project}) = 1 - \pi$$

$$P(r/\pi | r/\pi, \text{ safe project}) = P(r/\pi | r/\pi, \text{ risky project}) = 1.$$

The transition probability structure for interest rate arises from the following considerations.

(1) Given he has successfully paid back up to period t, by choosing the safe project BG-borrower can make sure that he can borrow in the next period at the interest rate $r_{t+1}^*$.

(2) Given he has successfully paid back up to period t, the risky project generates two possibilities. If it succeeds (with probability $\pi$), the interest rate at period t+1 is $r_{t+1}^*$. If it fails (with probability $1-\pi$), the lenders believe that the borrower is type B. Hence the lenders will charge the interest rate $r/\pi$.

(3) Once the borrower has defaulted so that the lenders believe that the borrower is of type B, the lenders' belief is unshakable; i.e., they believe forever that the borrower is of type B, therefore the lenders will charge the interest rate $r/\pi$ forever.

First of all, we will show that as $\delta$ goes to one, a lower bound for equilibrium payoff for BG-borrower approaches $G-r$. The intuition behind this lower bound is following. One feasible strategy for BG-borrower is to choose the safe project in every period. By doing this, the borrower can successfully convince the lenders (to any desired degree of assurance short of certainty) that he is of type G, therefore eventually he can borrow at interest rate arbitrarity close to r. Hence he can guarantee himself a payoff that approaches $G-r$ as $\delta \to 1$.

[6] From now on, we assume that $B \geq r/\pi$ so that even after the default, the market does not fail. We can easily incorporate the case where the market fails after the default. If we interpret the case where the market fails as the interest rate charged by the lenders to be $\infty$, we only need to replace $r/\pi$ with $\infty$ in the transition probability.

**Proposition 3.** Let $V_{BG}(\delta)$ be the equilibrium payoff the BG-borrower at discount rate $\delta$.[7] Under assumptions (1) and (3),

$$\liminf_{\delta \to 1} V_{BG}(\delta) \geqslant G - r.$$

Proof: See Appendix B

In proving proposition 3, we do not assume that $G-r>\max\{0, \pi(B-r/\pi)\}$. We need this assumption in order to prove the counterpart of Proposition 3 which asserts that $\limsup_{\delta \to 1} V_{BG}(\delta) \leqslant G-r$.

Since BG-borrower has zero probability, once a borrower defaults, the lenders believe that the borrower is of type B. Hence after a default, the equilibrium interest is $r/\pi$. Therefore the continuation payoff per period is $v^0 = \pi(B-r/\pi)$, if ever the borrower defaults.

Since the equilibrium interest rate given by equation (4) is monotonically decreasing to r, the best BG-borrower can do at given discounting factor $\delta$ is at most what he would get at the same $\delta$ if interest rate is r until he defaults once, thereafter $r/\pi$ is charged by the lenders. We will show that $G-r$ is an upper bound for the latter case and therefore automatically an upper bound for the original case.

Let $V^*_{BG}(\delta)$ be the Nash equilibrium payoff for BG-borrower in infinite horizon model if interest rate is r until default. For all $\delta$, $V_{BG}(\delta) \leqslant V^*_{BG}(\delta)$. Below we will show that for all $\delta$ sufficiently close to 1, $V^*_{BG}(\delta) = G-r$ so that $V_{BG}(\delta) \leqslant G-r$, for all $\delta$ sufficiently close to 1, from which $\limsup_{\delta \to 1} V_{BG}(\delta) \leqslant G-r$ follows.

**Lemma 2.** For all $\delta$ sufficiently close 1, $V_{BG}(\delta) = G-r$.

Proof: BG-borrower faces the stationary discounted dynamic programming problem with the following transition probability structure for interest rates.

$P(r|r, \text{safe project}) = 1.$

$P(r|r, \text{risky project}) = \pi$, and $P(r/\pi|r, \text{risky project}) = 1-\pi.$

$P(r/\pi|r/\pi, \text{safe project}) = P(r/\pi|r/\pi, \text{risky project}) = 1.$

We will show that the following strategy is unimprovable hence optimal: Choose the safe project unless there has been an earlier default. If ever there is a default, choose the risky project. In the first period, choose the safe project.

[7] There may be multiple equilibrium strategies. But the equilibrium payoff is unique.

By following the above strategy, the BG-borrower obtains $G-r$. By deviating to the risky project in one period and returning to the above strategy, he gets $(1-\delta)\pi(B-r)$ as the current period payoff. If the risky period succeeds (with probability $\pi$), the continuation value is $G-r$, if it fails (with probability $1-\pi$), the continuation payoff is $\upsilon^0$ as defined above. Therefore the one shot deviation to the risky project gives the BG-borrower the payoff

$$(1-\delta)\pi(B-r)+\delta[\pi(G-r)+(1-\pi)\upsilon^0].$$

Since $G-r>\upsilon^0$, for all $\delta$ sufficiently close to 1, the strategy of choosing the safe project unless default is an unimprovable strategy, therefore an optimal strategy. The payoff from an optimal strategy is $G-r$, which shows that $V^*_{BG}(\delta)=G-r$ for all $\delta$ sufficiently close to 1. Q.E.D.

By virtue of lemma 2, we can prove proposition 4.

**Proposition 4.** Under assumptions (1) and (3),

$$\limsup_{\delta\to 1} V_{BG}(\delta)\leqslant G-r.$$

Proof: This follows easily from lemma 2. Since $V_{BG}(\delta)\leqslant V^*_{BG}(\delta)$, for all $\delta$, by lemma 2, $V_{BG}(\delta)\leqslant G-r$ all $\delta$ sufficiently close 1. Hence $\lim\sup_{\delta\to 1} V_{BG}(\delta)\leqslant G-r$ as desired. Q.E.D.

By proposition 3 and 4, we can prove that $\lim_{\delta\to 1} V_{BG}(\delta)=G-r$. We record this as theorem 2.

**Theorem 2.** Under the assumptions (1) and (3), the set of Nash equilibrium payoff for BG-borrower converges to $G-r$ as $\delta$ goes to 1.

### *B. Infinite Horizon Model with $P_{bg}>0$*

In this subsection, we consider the full model where each type of borrower has strictly positive prior probability. In this case, we have to consider the interaction between the equilibrium interest rate and the project choice of BG-borrower. Despite the seeming increase in complexity, we can show more easily that the result in previous subsection still holds.

Since the case where the market fails from the beginning is uninteresting, we assume

$$\text{(5)} \qquad r^*=\frac{r}{P_g+\pi(P_{bg}+P_b)}<G.$$

Even if in equilibrium BG-borrower chooses the risky project at first period, the equilibrium interest is r* which is less than G by (5), therefore the G-borrower can borrow at that interest rate. So there is no separation between G and B or BG-borrower from the beginning. Similarly in previous subsection, first of all, we will show that $G-r$ is a lower bound for equilibrium payoff for BG-borrower, as $\delta$ goes to 1. Even though the project choice by BG-borrower influences the interest rate, the intuition is the same as the previous case. Namely, one feasible strategy for BG-borrower is to choose safe project every period. By doing this, he can successfully convince the lenders that he is of type G, therefore eventually he can borrow at interest rate r. Hence he can guarantee himself $G-r$.

**Proposition 5.** Under assumptions (1) and $(5)_2$

$$\liminf_{\delta \to 1} V_{BG}(\delta) \geqslant G-r.$$

Proof: See Appendix C.

Just as in the previous subsection, we do not assume that $G-r > \max\{0, \pi(B-r/\pi)\}$ in proving proposition 5. We need this assumption in order to show that $G-r$ is also an upper for Nash equilibrium payoff for BG-borrower as $\delta$ goes to 1.

In previous subsection, in proving proposition 4, we heavily counted on lemma 2. But in this subsection, we don't need this lemma. Instead we use the following observation that under assumption (1), in any equilibrium, the sum of the expected payoff to lenders and borrower does not exceed $(P_g + P_{bg})G + P_b \pi B$.

**Proposition 6.** Under assumptions of (1) and (5),

$$\limsup_{\delta \to 1} V_{BG}(\delta) \leqslant G-r.$$

Proof: First of all, note that (1) implies $G > \pi B$. Hence in expected terms, the safe project produces more surplus than risky project does.

G-borrower produces G by the safe project. BG-borrower produces either G or $\pi B$ in expected terms by the safe and risky project, respectively. Since $G > \pi B$ by (1), what BG-borrower can produce is at most G in expected term. B-borrower produce $\pi B$ in expected term by risky project. Hence the sum of the expected payoffs cannot exceed $(P_g + P_{bg})G + P_b \pi B$.

Let $V_L(\delta)$, $V_G(\delta)$, $V_{BG}(\delta)$ and $V_B(\delta)$ be equilibrium payoffs for lenders, G, BG and B-borrower, respectively in a fixed Nash equilibrium. Then

$$V_L(\delta)+P_gV_G(\delta)+P_bV_B(\delta)+P_{bg}V_{BG}(\delta)\leqslant(P_g+P_{bg})G+P_b\pi B.$$

Since we assume that lenders are competitive, $V_L(\delta)=r$, regardless of $\delta$, therefore we have

$$P_gV_G(\delta)+P_bV_B(\delta)+P_{bg}V_{BG}(\delta)\leqslant(P_g+P_{bg})(G-r)+P_b\pi(B-\frac{r}{\pi}.$$

Note that proposition 5 also asserts that $G-r$ is also a lower bound for Nash equilibrium payoff for G-borrower as $\delta$ goes to 1 hence we have $\lim\inf_{\delta\to 1} V_G(\delta)\geqslant G-r$.

For B-borrower, since $\frac{r}{\pi}$ is the worst possible interest rate we have

$$V_B(\delta)\geqslant(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi(B-\frac{r}{\pi}).$$

Therefore $\lim\inf_{\delta\to 1} V_B(\delta)\geqslant\pi(B-r/\pi)$.

Combining these inequalities we have

$$\begin{aligned}(P_g+P_{bg})(G-r)+P_b\pi(B-\frac{r}{\pi})&\geqslant\limsup_{\delta\to 1}\{P_gV_G(\delta)+P_bV_B(\delta)\\ &+P_{bg}V_{BG}(\delta)\}\geqslant P_{bg}\limsup_{\delta\to 1}V_{BG}(\delta)+P_g\liminf_{\delta\to 1}V_G(\delta)\\ &+P_b\liminf_{\delta\to 1}V_B(\delta)\geqslant P_{bg}\limsup_{\delta\to 1}V_{BG}(\delta)+P_g(G-r)\\ &+P_b\pi(B-\frac{r}{\pi}).\end{aligned}$$

Finally we have

$$P_{bg}\limsup_{\delta\to 1}V_{BG}(\delta)\leqslant P_{bg}(G-r).$$

Since $P_{bg}>0$, $\lim\sup_{\delta\to 1} V_{BG}(\delta)\leqslant G-r$. Q.E.D.

By propositions 5 and 6, we can prove that the set of Nash equilibrium payoff for BG-borrower converge to $G-r$ as $\delta$ goes to 1. We record this result as theorem 3.

**Theorem 3.** Under the assumptions (1) and (5), as $\delta\to 1$ the set of the equilibrium payoffs for BG-borrower converge to $G-r$.

By theorem 3, we can say that reputation effect works in infinite horizon model at least in terms of payoffs. Thus, the social efficiency is restored.

The result in this subsection is stronger than that in previous subsection in the following sense. In previous subsection, the equilibrium interest rate is determined independently of the project choice by BG-borrower. For BG-borrower, what he will do is to maximize his objective function given the anticipated equilibrium interest rate. This is just a single person decision problem so that the maximized value of objective function is exactly the Nash equilibrium payoff for BG-borrower, therefore the Nash equilibrium payoff is unique. What we have shown in previous subsection is such that this unique Nash equilibrium payoff converge to $G-r$ as $\delta$ goes to 1. Generally, the infinite horizon game admits multiple equilibria so that there may be multiple equilibrium payoffs. What we have shown in this subsection is that the set of Nash equilibrium payoff which may not be singleton for a fixed discount factor converge to $G-r$, as $\delta$ goes to 1. In other words, reputation works as a means for precommitment at least in terms of payoffs.

## Appendix A

In this appendix, we prove theorem 1 in several steps.

*Step 1.* First of all, we will prove the theorem for $\delta=1$ where the payoff is the undiscounted sum of payoffs at each period; i.e., $\Sigma_{t=1}^{T}$ (payoff at time t). Since for fixed period T, the series $\Sigma_{t=1}^{T} \delta^t a_t$ is continuous in $\delta$, our result follows.

Since $G-r>\max\{0, \pi(B-r/\pi)\}$, let $\sigma$ be such that $G-(r+\sigma)>\max\{0, \pi(B-r/\pi)\}$. For this $\sigma$, there exists $\mu$ such that if the marginal probability of safe project being chosen is $1-2\mu$, or more, then the interest rate is at most $r+\sigma$. Explicitly, if $r=(1-2\mu^*+2\mu^*\pi)(r+\sigma)$, then we get $\mu^*=\sigma/2(1-\pi)(r+\sigma)$, hence if $\mu\leqslant\mu^*$, the fact that the probability of safe project being chosen is at least $1-2\mu$ implies that interest rate is at most $r+\sigma$.

*Step 2.* The nex step is to find a $K_1$ such that if the borrower pays back for $K_1$ periods, the posterior probability for B-borrower is at most $\mu$, whatever BG-borrower does. The posterior probability for B-borrower given that the borrower pays back for K periods is given by

$$\frac{\pi^K P_b}{\pi^K P_b+\text{Prob(BG-borrower pays back K times)}+P_g}$$

One upper bound for above probability is given by $\pi^K/P_g$. As long as $\pi<1$ and $P_g>0$, as K goes to $\infty$, $\pi^K/P_g$ goes to zero. We choose $K_1(P_g)$ the samllest K such that $\pi^K/P_g \leqslant \mu$. By construction, regardless of what BG-borrower does, paying back for $K_1$ or more periods implies that the posterior probability for B-borrower is at most $\mu$. Note that $K_1(P_g)$ depends on $P_g$, not $P_b$ or $P_{bg}$ so that $K_1(P_g)$ works for all $\tilde{P}_g \geqslant P_g$.

*Step 3.* The next step is to find K* such that if the borrower pays back K* periods or more in which the probability of choosing safe projects by BG-borrower is at most $1-\mu$, then the posterior probability for BG-borrower given this information is at most $\mu$.

If BG-borrower chooses safe project with probability at most $1-\mu$, then one upper bound for the posterior probability for BG-borrower after K* periods of successful pay-back is given by

$$\frac{\{(1-\mu)+\mu\pi\}^{K^*}}{P_g}.$$

Since $\pi<1$, $(1-\mu)+\mu\pi<1$ and $P_g>0$, we choose $K^*(P_g)$ the smallest K* such that

$$\frac{\{(1-\mu)+\mu\pi\}^{K^*}}{P_g} \leqslant \mu$$

Hence, if the lenders observe that the borrower has paid back successfully for $K^*(P_g)$ or more in which BG-borrower is supposed to choose safe project with probability at most $1-\mu$, they conclude that the posterior probability of type BG is at most $\mu$.

Again, note that $K^*$ $(P_g)$ depends on $P_g$, not $P_b$ or $P_{bg}$ and that $K^*(P_g)$ works for all $\tilde{P}_g \geqslant P_g$.

*Step 4.* We will show that in any Nash equilibrium, in every subgame where T is the number of periods to go and the posterior probability for G-borrower is at least $P_g$, the prior probability for G-borrower, the equilibrium payoff for BG-borrower, V is at least $(T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$.

Let $K_2$ be the number of periods in equilibrium in which BG-borrower chooses the safe project with probability at most $1-\mu$ when T periods remain. We consider two cases.

*Case 1.* $K_2 \leqslant K^*(P_g)$.

One available strategy for BG-borrower is to choose safe project always. Suppose that BG-borrower deviates from the equilibrium strategy to the strategy of choosing safe project always, thereby paying back in every period. Since the probability G-borrower is at least $P_g$, $K_1(P_g)$ works so that after $K_1(P_g)$ or more periods of successful pay back, the posterior probability for B-borrower is at most $\mu$.

Along the equilibrium, the lenders expect that BG-borrower will choose the safe project with probability at most $1-\mu$ for $K_2$ periods so that they anticipate that at least for $T-K_1(P_g)-K_2$ periods, the safe project is chosen with probability at least $1-2\mu$. Therefore the interest rate is at most $r+\sigma$ for $T-K_1(P_g)-K_2$ periods or more.

By deviating to the strategy of choosing safe project always, BG-borrower gets at least $G-(r+\sigma)$ in $T-K_1(P_g)-K_2$ periods, gets no worse than zero in $K_1(P_g)+K_2$ periods. Since $K_2 \leqslant K^*(P_g)$, the continuation payoff, V along the equilibrium path when T periods remain is at least $(T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$.

*Case 2.* $K_2 > K^*(P_g)$.

Suppose that BG-borrower deviates to the strategy of choosing safe project always, thereby paying back in every period.

By the same argument in case 1, BG-borrower gets $G-(r+\sigma)$ or more in $T-K_1(P_g)-K_2$. Besides, since the probability for G-borrower is at least $P_g$, $K^*(P_g)$ works so that after the successful pay back for $K^*(P_g)$ or more, the posterior probability for BG-borrower is at most $\mu$. Since $K_2 > K^*(P_g)$, among the remaining $K_1(P_g)+K_2$ periods, at least for $K_2-K^*(P_g)$ periods, the probability for B or BG-borrower is at most $2\mu$. Therefore, whatever BG-borrower does, the probability of safe project be chosen is at least $1-2\mu$ so that the interest rate is at most $r+\sigma$. Hence BG-borrower gets at least $G-(r+\sigma)$ for $K_2-K^*(P_g)$ periods.

By adding up these two payoffs, BG-borrower gets at least $(T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$. Hence V, the continuation payoff along the equilibrium path when T periods remain is at least $(T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$.

In either case, in the subgame where the remaining periods to go are T and the probability for G-borrower is at least $P_g$, the continuation payoff for BG-borrower along the equilibrium path is at least $(T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$.

*Step 5.* Now think of the choice problem of BG-borrower at time $T+1$, where the probability for G-borrower is at least $P_g$. Let V be the continuation payoff when he can pay back at period $T+1$. Since the posterior probability for G-borrower after a successful payback cannot decrease, after a successful payback at period $T+1$, the posterior probability for G-borrower is at least $P_g$. Hence by step 4, $V \geqslant (T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$.

By choosing safe project, BG-borrower gets no where than zero at period $T+1$ and gets V as a continuation payoff. Therefore, he gets at least V. By choosing risky project he gets at most $(B-r)$ at period $T+1$. If the risky project succeeds so that he can pay back at period $T+1$, he gets V as the continuation value. If it fails, however, he is revealed to be of type B or BG, hence by Proportion 2, his continuation payoff is $T[\max\{0, \pi(B-r/\pi)\}]$. Since $\pi$ is the probability of failure, the continuation payoff is $\{\pi V+(1-\pi)T[\max\{0, \pi(B-r/\pi)\}]\}$. Therefore by choosing risky project, BG-borrower gets at most $(B-r)+\{\pi V+(1-\pi)T[\max\{0, \pi(B-r/\pi)\}]\}$.

Define H be the difference between what the type BG borrower gets by choosing the safe project and what he does by choosing the risky project. We will show that H is strictly positive so that the BG-borrower indeed chooses safe project at period $T+1$.

$$
\begin{aligned}
H &\geqslant V-\pi(B-r)-[\pi V+(1-\pi)T\{\max\{0, \pi(B-\frac{r}{\pi})\}\}] \\
&=(1-\pi)V-(1-\pi)T\{\max\{0, \pi(B-\frac{r}{\pi})\}\}-(B-r) \\
&\geqslant(1-\pi)[T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))-T\{\max\{0, \pi(B-\frac{r}{\pi})\}\}] \\
&\quad -\pi(B-r)
\end{aligned}
$$

The second inequality follows from the fact that $V \geqslant (T-K_1(P_g)-K^*(P_g))(G-(r+\sigma))$. Since $G-(r+\sigma)>\max\{0, \pi(B-r/\pi)\}$, we can choose $T^*$ as the smallest integer such that for all $T \geqslant T^*$, $H>0$. Hence at time period $T^*+1$, BG-borrower chooses the safe project.

We have shown that in every subgame where the probability of type G is at least $P_g$, the prior probability and the number of periods to go is greater than $T^*$, the type BG borrower chooses the safe project. Therefore by induction, BG-borrower always chooses the safe project except possibly last $T^*$ periods.

## Appendix B

In this appendix, we prove Proposition 3.

Let $\{\sigma^*(t)\}_{t=1}^{\infty}$ denote part of an equilibrium strategy for BG-borrower in a Nash Equilibrium, where $\sigma^*(t)$ is a probability of choosing safe project at period t if he has paid back successfully so far.

Note that the equilibrium interest rate along the successful payback path is given by equation (4) so that by deviating to choosing safe project BG-borrower can guarantee himself.

$$H(\delta)=(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}(G-r_t^*),\ \text{where } r^*_t=r\frac{\pi^{t-1}P_b+P_g}{\pi^t P_b+P_g}.$$

By rearranging the terms in $H(\delta)$, we get

$$H(\delta)=G-(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}r^*_t$$

$$=G-(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}r\left(\frac{\pi^t P_b+P_g}{\pi^{t-1}P_b+P_g}\right)=G-(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}$$

$$\left(1+\frac{P_b\pi^{t-1}(1-\pi)}{P_b\pi^t+P_g}\right)r=G-r-(1-\delta)r(1-\pi)\sum_{t=1}^{\infty}\frac{P_b(\delta\pi)^{t-1}}{P_b\pi^t+P_g}.$$

Next step is to show that the series, $\Sigma_{t=1}^{\infty}\frac{P_b(\delta\pi)^{t-1}}{P_b\pi^t+P_g}$ converges for all $0\leqslant\delta\leqslant 1$. For this purpose we apply the ratio test to the series.

Let $a_t(\delta)=\frac{P_b(\delta\pi)^{t-1}}{P_b\pi^t+P_g}$, then

$$\frac{a_{t+1}(\delta)}{a_t(\delta)}=\frac{\dfrac{P_b(\delta\pi)^t}{P_b\pi^{t+1}+P_g}}{\dfrac{P_b(\delta\pi)^{t-1}}{P_b\pi^t+P_g}}$$

$$=\delta\pi\,\frac{P_b\pi^t+P_g}{P_b\pi^{t-1}+P_g}.$$

Therefore

$$\lim_{t\to\infty}\frac{a_{t+1}(\delta)}{a_t(\delta)}=\delta\pi<1.$$

Even if $\delta=1$,

$$\lim_{t\to\infty}\frac{a_{t+1}(1)}{a_t(1)}=\pi<1.$$

Hence we know that for all $0\leqslant\delta\leqslant 1$,

$\sum_{t=1}^{\infty} a_t(\delta)$ converges.

Therefore as $\delta$ goes to one, $H(\delta)$ goes to $G-r$. Since $V(\delta) \geqslant H(\delta)$,

$$\liminf_{\delta \to 1} V_{BG}(\delta) \geqslant \lim_{\delta \to 1} H(\delta) = G - r, \text{ as desired. Q.E.D.}$$

## Appendix C

In this appendix, we prove proposition 5.

Let $\{\sigma^*(t)\}_{t=1}^{\infty}$ be part of an equilibrium strategy for BG-borrower in a Nash equilibrium, where $\sigma^*(t)$ is the probability of choosing the safe project at period t if the borrower has paid back successfully up to period $t-1$. Let $V_{BG}(\delta)$ be the equilibrium payoff for type BG associated with the overall equilibrium strategy. We will show that $\liminf_{\delta \to 1} V_{BG}(\delta) \geqslant G - r$.[8]

Let $\{r_t\}_{t=1}^{\infty}$ be the sequence of equilibrium interests along the history of successful payback. If BG-borrower deviates by choosing safe project at every period, the payoff is $H(\delta) = (1-\delta)\Sigma_{t=1}^{\infty} \delta^{t-1}(G - r_t^*)$. Hence we have

$$V_{BG}(\delta) \geqslant H(\delta) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}(G - r_t).$$

We will calculate a lower bound for $H(\delta)$ which is also a lower bound for $V_{BG}(\delta)$.

Note that we can calculate $r_t$ from the prior probabilities of each type and equilibrium strategy $\{\sigma^*(t)\}_{t=1}^{\infty}$. The probability for successful payback for $t-1$ period is given by 1, $\pi^{t-1}$ and $\Pi_{i=1}^{t-1}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}P_{bg}$ for G, B and BG-borrower, respectively. At period t the G-borrower can surely pay back, B-borrower can pay back with probability $\pi$, and since type BG borrower follows strategy $\sigma^*(t)$, $\{\sigma^*(t) + (1-\sigma^*(t))\pi\}$ is the probability that BG-borrower can pay back. Therefore the probability for payback at period t given that the borrower has paid back up to $t-1$ is given by

$$\text{Prob}\{\text{pay back at period t}|\text{pay back up to } t-1\} = \frac{P_g + \pi^t P_b + \Pi_{i=1}^{t}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}P_{bg}}{P_g + \pi^{t-1} P_b + \Pi_{i=1}^{t-1}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}P_{bg}}$$

[8] To be very precise, we would write $\sigma^*(t;\delta)$ for an equilibrium strategy parametrized by $\delta$. But we will be sloppy and suppress this dependence.

Hence $r_t$ is given by

$$(6) \qquad r_t = r\,\frac{P_g + \pi^{t-1}P_b + \Pi_{i=1}^{t-1}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}P_{bg}}{P_g + \pi^{t}P_b + \Pi_{i=1}^{t}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}P_{bg}}.$$

First of all we will show that $r_t$ converge to r as t goes to $\infty$. For this, define a sequence $\{b_t\}$ as follows:

$$b_t = \prod_{i=1}^{t}\{\sigma^*(i) + (1-\sigma^*(i))\pi\}.$$

Then $b_t = b_{t-1}\{\sigma^*(t) + (1-\sigma^*(t))\pi\}$. Since $\{\sigma^*(t) + (1-\sigma^*(t))\pi\} \leqslant 1$, $\{b_t\}$ is monotonously decreasing and bounded below by 0, hence converges to, say, b.

Then as t goes to $\infty$, we have

$$\lim_{t\to\infty} r_t = \lim_{t\to\infty} r\,\frac{P_g + \pi^{t-1}P_b + b_{t-1}P_{bg}}{P_g + \pi^{t}P_b + b_tP_{bg}} = r\,\frac{P_g + bP_{bg}}{P_g + bP_{bg}} = r.$$

Hence given $\epsilon > 0$, there exists $N(\epsilon)$ such that for all t $N(\epsilon)$, $r_t^* \leqslant r + \epsilon$.

Now we calculate a lower bound for $H(\delta)$.

$$\begin{aligned}
H(\delta) &= (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}(G - r_t^*) = G - (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}r_t^* \\
&= G - (1-\delta)\sum_{t=1}^{N(\epsilon)}\delta^{t-1}r_t^* - (1-\delta)\sum_{t=N(\epsilon)+1}^{\infty}\delta^{t-1}r_t^* \\
&\quad G - (1-\delta)\sum_{t=1}^{N(\epsilon)}\delta^{t-1}r_t^* - (1-\delta)\sum_{t=N(\epsilon)+1}^{\infty}\delta^{t-1}(r+\epsilon) \\
&= G - (1-\delta)\sum_{t=1}^{N(\epsilon)}\delta^{t-1}r_t^* - (r+\epsilon)\delta^{N(\epsilon)}.
\end{aligned}$$

Therefore, finally, we have

$$V_{BG}(\delta) \geqslant H(\delta) \geqslant G - (1-\delta)\sum_{t=1}^{N(\epsilon)}\delta^{t-1}r_t^* - (r+\epsilon)\delta^{N(\epsilon)}.$$

Hence as $\delta$ goes to 1, we have

$$\begin{aligned}
&\liminf_{\delta-1} V_{BG}(\delta) \geqslant \lim_{\delta-1} G - (1-\delta)\sum_{t=1}^{N(\epsilon)}\delta^{t-1}r_t - (r+\epsilon)\delta^{N(\epsilon)} \\
&= (G - r) - \epsilon.
\end{aligned}$$

Since $\epsilon$ is arbitrary, we have $\liminf_{\delta-1} V(\delta) \geqslant G - r$. Q.E.D.

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