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A Modified Version of Breeden's Capital Asset Pricing Model

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The market-based model, developed by Sharpe and Lintner, asserts that the risk premium is proportional to the covariance of the rate of return with the rate of return on the market portfolio. Recently Breeden has put forward a consumption-based model as an alternative: he replaces the rate of return on the market portfolio with aggregate consumption.

Using the permanent-income theory of consumption demand, we modify the Breeden's model as a permanent-consumption based CAPM. We show that the rate of return on the market portfolio must be replaced by the permanent consumption, not by the actual consumption.

I. Introduction

The market-based model, developed by Sharpe and Lintner, asserts that the risk premium is proportional to the covariance of the rate of return with the rate of return on the market portfolio. Recently Breeden has put forward a consumption-based model as an alternative: he replaces the rate of return on the market portfolio with aggregate consumption.

Many investigators have tested the standard CAPM, but they have furnished little econometric evidence that the model is actually valid. Roll argues that the econometric tests of the traditional model prove little because one cannot accurately measure the rate of return on the market portfolio. Breeden argues that

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assume that the instantaneous rate of return da is generated by an Itô process;

$$(1)\ da = \mu\, dt + dz.$$

Here μ is an n-vector; dz is an n-vector of stationary Wiener processes such that with $E_t(dz) = 0$ and $Var_t(dz) = S$. The instantaneous rates of return vector thus has mean μ and variance S.

Given his wealth, at each instant each individual chooses an optimal rate of consumption and optimal portfolio. Wealth is w and consumption is c. The n-vector f describes the portfolio. The component f_i denotes the fraction of the total wealth invested in the risky asset *i*. The remainder of the wealth is invested in the riskless asset. Given this framework, the investor's wealth change can be given by the stochastic differential equation;

$$(2)\ dw = (1 - f'1_n)\, wr\, dt + f'w\, da - c\, dt.$$

Here $(1 - f'1_n)$ is the fraction of the total wealth invested in riskless asset; 1_n is an n-vector of ones. Substituting (1) for da, we can rewrite (2) as

$$(3)\ dw = (f'(\mu - r1_n) + r)w\, dt + wf'\, dz - c\, dt.$$

The choice of $f = (f_1, f_2, f_3, \ldots, f_n)'$ is unconstrained, because the fraction $(1-f'1_n)$ of the total wealth is invested in riskless asset.

We assume that each individual *k* acts to maximize the expected value of his time-additive, state-independent, strictly concave *Von Neumann-Morgenstern* utility function;

$$(4)\ \max_{c,f} E_t\left\{\int_t^\infty U[c(\tau), \tau]\, d\tau\right\},$$

subject to the budget constraint (3). Here E_t is the conditional expectation operator, conditional on the information at time *t*.

We can derive the necessary optimality conditions for an investor facing the problem (4). First let us define the valuation function $V[w(t), t]$ as follows;

The proportion invested in each is determined by the relative risk aversion. The investment in the efficient portfolio of risky assets decreases as the risk aversion increases.

A. Market-Based CAPM

Now we consider market equilibrium. Let the superscript k denote the value for individual k. The total demand for risky assets is $\sum_k f^k w^k$. We assume homogeneous expectations; all individuals have the same expectations.

The market portfolio refers to the portfolio of all outstanding assets. The n-vector f describes the market portfolio; f_i is the fraction of the total wealth in asset i. Consequently, $1 - f'1_n$ is the fraction of total wealth in the riskless asset. The rate of return on the market portfolio is

$$(10)\ da_m = f'\,da + (1 - f'1_n)r\,dt$$
$$= (f'(\mu - r1_n) + r)dt + f'\,dz.$$

The market rate of return has mean $f'(\mu - r1_n) + r$ and variance $f'Sf$.

Equilibrium requires that supply equals demand:

$$(11)\ fw = \sum_k f^k w^k.$$

Hence substituting f^k from (9) gives the market portfolio:

$$(12)\ f = [\sum_k (1/R_R{}^k) w^k/w] S^{-1}(\mu - r1_n) = (1/R_R) S^{-1}(\mu - r1_n),$$

in which we define $1/R_R = \sum_k (1/R_R{}^k) w^k/w$. Thus R_R is the harmonic mean of the relative risk aversion across individuals in the sense that R_R is the weighted average of individual relative risk aversion with weights w^k. Equivalently,

$$(13)\ \mu = r1_n + R_R\,Sf = r1_n + R_R\,Cov_t(da, da_m),$$

Premultiplying (15) by S and using the $\mathrm{Cov}_t(da, dw) = wf'S$ by (1) and (3), we can rewrite (15) as

$$(16)\ \mu - r1_n = -(U_{cc}c_w/U_c)\mathrm{Cov}_t(da, dw),$$

where $\mathrm{Cov}_t(da, dw)$ is the n-vector of covariances of asset returns with the wealth change. Since the optimal consumption is a function of wealth and time c(w, t), Itô's lemma implies $dc = c_w\, dw + c_t\, dt + (1/2)c_{ww}\, (dw)^2$.

Therefore

$$(17)\ \mathrm{Cov}_t(da, dc) = \mathrm{Cov}_t(da, dw)c_w.$$

Intuitively (17) can also be seen by noting that the random change in consumption rate is linear in the random change in wealth, with the weight in the linear relation being the partial derivatives of the consumption with respect to the wealth. Thus the covariance of asset returns with the change in consumption can be expressed by (17). We rewrite (16) as

$$(18)\ \mu - r1_n = -(U_{cc}/U_c)\ \mathrm{Cov}_t(da, dc).$$

We can readily aggregate this relationship. Let $c = \sum_k c^k$ denote aggregate consumption. We have

$$(19)\ \mathrm{Cov}_t(da, dc^k) = -(U_c/U_{cc})\ (\mu - r1_n).$$

Summing across all individuals,

$$(20)\ \mathrm{Cov}_t(da, dc) = -[\Sigma_k(U_c/U_{cc})^k]\ (\mu - r1_n).$$

So

$$(21)\ \mu - r1_n = [-1/\Sigma_k(U_c/U_{cc})^k]\ \mathrm{Cov}_t\ (dt, dc)$$

$$= [-c/\Sigma_k c^k(U_c/cU_{cc})^k]\ \mathrm{Cov}_t(da, dc/c).$$

Thus $-c/\sum_k c^k(U_c/cU_{cc})^k$ is harmonic mean of $-c^k U_{cc}{}^k/U_c{}^k$.

Therefore

$$(22)\ \mu - r1_n = -(cU_{cc}/U_c)\ \mathrm{Cov}_t(da, dc/c),$$

consumer to the problem of dividing consumption between the present and the future. According to this hypothesis, consumers form estimates of their ability to consume in the long run and then set permanent consumption to their appropriate fraction of that estimate. The estimate may be stated in the form of wealth, following Modigliani, or permanent income, following Friedman. Therefore we have

$$(24)\ c_p \propto w,$$

where c_p denotes the permanent consumption and w is the wealth.[2]

Since stochastic parts of da_m and dw/w are the same, we have $Cov_t(da, da_m) = Cov_t(da, dw/w)$. Also, by (24), we have $Cov_t(da, dw/w) = Cov_t(da, dc_p/c_p)$. Using these, we can rewrite (13) as;

$$(25)\ \mu - rl_n = R_R\ Cov_t\ (da, dc_p/c_p)$$
$$= R_R\ Cov_t\ (da, dln\ c_p),$$

where dln c_p denotes the change in permanent consumption. In the permanent consumption-based model, the permanent consumption serves as a sufficient statistic for the underlying wealth. The risk premium depends on the covariance of the rate of return with permanent consumption.

IV. Summary and Further work

Recent advances in theoretical finance have suggested a fresh way of looking at the relationship between the expected return and risk. Especially Breeden restates the market-based model as a consumption-based model. We show how Breeden does develope his model in a continuous-time framework. Also we show that his

[2] For instance, it will be reasonable assumption that investors take time to adjust the change of wealth. Then the change of wealth can affect the future consumption as well as current consumption. In that case, we can make hypothesis that the change in permanent consumption may be expressed as a distributed-lead function of the current and future changes in actual consumption. Therefore the change of wealth can be replaced by the change of permanent consumption, as implied by (24).

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