A Theoretical Analysis of the Efficient Provision of a Public Good

Wankyu Park*

I. Introduction

The free-rider problem has long been regarded as intrinsically unsolvable as a practical manner. The essence of this problem is that for a Pareto-optimal solution to be reached, individuals must reveal their preferences for public goods. However since each individual consumes the total quantity of public goods supplied, it is in any individual's interest to either understate or overstate his preferences depending on how the stated preferences are used in the financing process of the public good. We are thus left without a direct way to determine the efficient level at which to produce a public good.

A number of recent papers have developed a variety of ways to overcome the free-rider problem. However the literature on the theoretical design of the mechanisms has not dealt with empirical issues, nor used real world data. The purpose of this paper is to analyze the efficient provision of a public good using the weak complementarity approach of Bradford and Hildebrandt (1977) and Hildebrandt and Tregarthen (1979-hereafter H-T).

This paper is organized as follows. In section II, it is shown that different individual behaviors (altruistic as opposed to self-interested behavior) toward financing a public good result in different amounts of willingnesses-to-pay for a public good. It is also

* Assistant Professor, Department of Economics, Chung-Ang University, Seoul, Korea.

$$\Rightarrow \int_{o}^{x^i} p(\xi^i, q, y^i)\, d\xi^i = \int_{p}^{\bar{p}} x^i(\tau, q, y^i) d\tau + \int_{\bar{p}}^{\infty} x^i(\tau, q, y^i) d\tau + x^i p$$

$$\Rightarrow \int_{o}^{x^i} p_q(\xi^i, q, y^i)\, d\xi^i = \int_{p}^{\bar{p}} x^i_q(\tau, q, y^i)\, d\tau + \int_{\bar{p}}^{\infty} x^i_q(\tau, q, y^i) d\tau .$$

Proposition 1: If the two restrictions mentioned above are satisfied, then

$$\int_{p}^{\infty} x^i_q(\tau, q, y^i) d\tau = 0 \text{ and } \int_{o}^{x^i} p_q(\xi^i, q, y^i)\, d\xi^i = \int_{p}^{\bar{p}} x^i_q(\tau, q, y^i) d\tau .$$

Proof:

(3) $x^i = - V^i_p / V^i_y$ (by Roy's identity),

(4) $x^i_q = - \partial (V^i_p / V^i_y)/\partial q = -(V^i_{pq} V^i_y - V^i_p V^i_{qy})/(V^i_y)^2 .$

By the restriction of interdependence,

(5) $V^i_q / V^i_y = 0$ at $p = \bar{p}$. Then $\partial (V^i_q / V^i_y)/\partial p = 0$ at $p = \bar{p}$.

From (5),

(6) $(V^i_{pq} V^i_y - V^i_q V^i_{py})/(V^i_y)^2 = 0$ at $p = \bar{p}$.

dividual has positive demand for a private good, a public good becomes valueless at a specific price level. However since individuals' altruistic behavior (or public regardedness behavior[1]) may have an important role in the provision of a public good, it is likely that an individual will still place some valuation on the public good at a threshold price of the private good. Therefore it is assumed here that an individual's marginal valuation of the public good stays constant with respect to the private good price at the threshold price level. This assumption satisfies assumption (1) as a special case. Specifically, these two restrictions are expressed as follows:

$$(8) \qquad \partial(V^i_q/V^i_p)/\partial y^i = 0\ ^2,$$

1 According to Hanushek (1975), some individuals are inclined to support public expenditures above those indicated by their own private interests and this public regardedness is linked to their heritage and to their ethnic background. Also according to Deacon (1977), the appearance of certain anomalous results in the study (e.g., high incomes voter favoring welfare proposals), seemingly at odds with narrow self-interest, led to the introduction of "public regardingness" as an attribute that influences the behavior of certain calsses of voters.

2 It can be shown that the usual willingness-to-pay expression in terms of the direct utility function ($=U_q/U_y$) is equivalent to the indirect utility expression ($=V_q/V_y$).

Let's consider the following constrained maximization problem:

$$\text{Max } U(x, q, N)$$
$$\text{s.t. } px + H = y,$$

where N is a numeraire goods. Let

$$(2\text{-}1) \qquad U(x,q,N) \equiv \overline{U}.$$

Total differentiation of (2-1) gives

$$(2\text{-}2) \qquad U_x dx + U_q dq + U_N dN = 0.$$

From the budget constraint (with p fixed),

$$(2\text{-}3) \qquad dN = dy - pdx.$$

Marginal utility of the numeraire good is equivalent to marginal utility of income, i.e.,

$$(2\text{-}4) \qquad U_N = U_y.$$

From (2-2), (2-3) and (2-4), we get

$$U_x dx + U_q dq + U_y(dy - pdx) = 0$$

$$(2\text{-}5) \qquad (U_x - pU_y)dx + U_q dq + U_y dy = 0.$$

Since efficiency requires that U_x/U_y be equal to p, the first term of expression (2-5) vanishes:

$$(2\text{-}6) \qquad -\frac{dy}{dq}\Big|_{U=\overline{U}} = \frac{U_q}{U_y}.$$

From the indirect utility function, let

$$(2\text{-}7) \qquad V(p, q, y) \equiv \overline{U}.$$

change in the private good demand due to change in the public good is as follows:

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} - \frac{V_q(p', q, y)}{V_y(p', q, y)} = \int_p^{p'} - \frac{\partial (V_q/V_y)}{\partial \tau} d\tau$$

Since $\partial (V_q/V_p)/ \partial y$ is equal to zero from assumption (2), $V_{qy} V_p$ is equal to $V_q V_{py}$. And

$$- \frac{\partial (V_q/V_y)}{\partial p} = - \frac{(V_{qp} V_y - V_q V_{yp})}{V_y^2} ,$$

$$\frac{\partial x}{\partial q} = - \frac{\partial (V_p/V_y)}{\partial q} = - \frac{(V_{pq} V_y - V_p V_{yq})}{V_y^2}$$

Therefore, $- \partial (V_q/V_y/) \partial p = \partial x/ \partial q$ from assumption (2):

$$\int_p^{p'} - \frac{\partial (V_q/V_y)}{\partial \tau} d\tau = \int_p^{p'} x_q(\tau, q, y) d\tau .$$

This implies that $\partial (V_q/V_y)/\partial p$ is equal to zero at $p = p'$ if and only if x_q is equal to zero at $p = p'$. With assumption (9), $\partial x/\partial q$ is equal to zero at $\bar{\bar{p}}$. On the other hand if assumption (1) holds, then assumption (9) always holds at $\bar{p}$, but not vice versa.

It is also noteworthy to compare the two assumptions, (1) and (9), based on the properties of an indirect utility function. Let's define complements first. If a private good demand (x) increases as a public good (q) increases, i.e., $\partial x/\partial q > 0$, then they are defined as complements in consumption.[3]

From the indirect utility function (10),

[3] This definition is different from one in Deaton and Muellbauer (1980, p. 46). Their definition is that goods i and j are complements if $\partial h_i/ \partial P_j$ is negative, and they are substitutes if $\partial h_i/ \partial P_j$ is positive, where h_i is the Hicksian (or compensated) demand.

curve is caused by the increase in the level of the public good provision (from q to q'). If the price of the private good is equal to $\bar{\bar{p}}$, then there will be no change in the private good demand; i.e., $x_q(p, q, y) = 0$ at $p = \bar{\bar{p}}$.

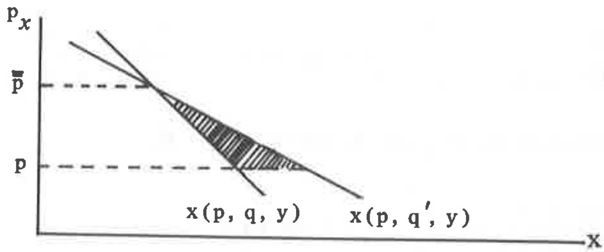

Figure 2

At a low price of the private good, demand for the good goes up as the provision of a public good increases ($x_q > 0$ at $p < \bar{\bar{p}}$). As the price goes up, the degree of increase in the private good due to the same increase in the public good goes down ($x_{qp} < 0$). Finally at a high price there will be no change in the private good demand due to the increase in the public good ($x_q = 0$ at $p = \bar{\bar{p}}$). And beyond that price the same increase in the public good reduces the demand for the private good. In this case incremental consumer's surplus can be maximized by setting the threshold price at $\bar{\bar{p}}$. This maximized surplus is drawn as the shaded area. Any other price will lower the incremental consumer's surplus. The threshold price level is chosen with assumption (9). The numerical value of the threshold price depends on the specification of the private good demand function. If the quantity demanded is zero at p' ($p' < \bar{\bar{p}}$), then threshold price must be p'. If the private good demand is of the form of a hyperbolic function, both V_q and V_{qp} becomes zero at an infinite price level and quantity demanded will be zero at this price level.

Proposition 2 (Self-Interested Behavior): With restrictions (1) and (2), an individual's willingness-to-pay is captured with the Marshallian private good demand function in the interval between p and $\bar{\bar{p}}$.

Proof: From Willing's Lemma 2 (1978, p. 241), Let $\mu(\bar{\bar{p}}, q \mid p^\circ, q^\circ, y^\circ) = y$. Then

$$(16) \qquad \mu_q(\bar{\bar{p}}, q \mid p^\circ, q^\circ, y^\circ) = -\frac{V_q(\bar{\bar{p}}, q, y)}{V_y(\bar{\bar{p}}, q, y)}$$

Willing's Lemma 5 (1978, p. 243) states that restriction (8) is equivalent to

$$(17) \qquad \mu_q(p, q) = -\int_p^\infty x_q\{\tau, q, \mu(p, q)\}d\tau \; .$$

Let p = p. Then

$$\mu_q(\bar{\bar{p}}, q) = -\int_{\bar{\bar{p}}}^\infty x_q\{\tau, q, \mu(\bar{\bar{p}}, q)\}\, d\tau = -\int_{\bar{\bar{p}}}^\infty x_q(\tau, q, y)d\tau \, .$$

Note that since $\mu(\bar{\bar{p}}, q \mid p^\circ, q^\circ, y^\circ)$ is defined as y, $x\{p, q, \mu(\bar{\bar{p}}, q)\}$ is the Marshallian private good demand. From Willig's Lemma 2,

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} = \int_p^{\bar{\bar{p}}} x_q(\tau, q, y)d\tau + \frac{V_q(\bar{\bar{p}}, q, y)}{V_y(\bar{\bar{p}}, q, y)}$$

$$= \int_p^{\bar{\bar{p}}} x_q(\tau, q, y)d\tau + \int_p^\infty x_q\{\tau, q, \mu(\bar{\bar{p}}, q)\}d\tau$$

$$= \int_p^{\bar{\bar{p}}} x_q(\tau, q, y)d\tau + \int_{\bar{\bar{p}}}^\infty x_q(\tau, q, y)d\tau$$

$$= \int_p^{\bar{\bar{p}}} x_q(\tau, q, y)d\tau \; .$$

Note that in this case, the relevant upper price limit will be $\bar{\bar{p}}$, because beyond $\bar{\bar{p}}$ the complementary relationship between the two goods will be changed and incremental consumer's surplus will be decreased. The efficiency condition will be

$$\sum_{i=1}^n \frac{V_q^i}{V_y^i} = \sum_{i=1}^n \int_p^{\bar{\bar{p}}} x_q^i(\tau, q, y^i)d\tau = MC_q.$$

$$\int_{\bar{p}}^{p} \frac{V_{pq}V_y - V_pV_{yq}}{V_y^2}\,d\tau = \int_{\bar{p}}^{p} \frac{V_{pq}}{V_y}\,d\tau$$

$$= \frac{1}{V_y}[V_q(p, q, y) - V_q(\bar{p}, q, y)] = \frac{V_q(p, q, y)}{V_y(p, q, y)}.$$

Therefore willingness-to-pay will be

$$\frac{V_q}{V_y} = - \int_{\bar{p}}^{p} x_q(\tau, q, y)d\tau .$$

Note that the substitutive relationship will be changed below $\bar{p}$. The threshold price $\bar{p}$ can be zero depending on the form of the private good demand function. The efficiency condition will be

$$\sum_{i=1}^{n} \frac{V_q^i}{V_y^i} = - \sum_{i=1}^{n} \int_{\bar{p}}^{p} x_q^i(\tau, q, y^i)d\tau = MC_q.$$

Up to now we have analyzed the implications behind an indirect utility function which satisfies two different sets of assumptions. According to this analysis, what we have to do first is to construct an indirect utility function which satisfies relevant assumptions. The next step is to calculate the threshold price either from the indirect utility function or from the private good demand function. The final step is to sum individual willingnesses-to-pay and to equate this with the marginal cost of a public good and to solve for the efficient level of the public good. However enumerating these steps in this way is misleading. The major advantage of the weak complementarity approach lies in the relatively small amount of information we need to solve for the efficient level of a public good. Once we can set up the direct (or indirect) utility function, we no longer have to worry about the willingness-to-pay for the public good. The essence of the weak complementarity approach is that what we need is a private good demand function which is not independent of a public good. With some restrictions on the private good demand (thus on the implied indirect utility function), we can solve for the efficient level of a public good. In this sense following proposition has an important role.

$$x(p,q,y) = x(p,q,y^\circ)\left(\frac{y}{y^\circ}\right)^\eta.$$

where y° is the base income. From Roy's identity,

$$x(p,q,y^\circ)\left(\frac{y}{y^\circ}\right)^\eta \frac{\partial V(p,q,y)}{\partial y} = -\frac{\partial V(p,q,y)}{\partial p}$$

New let's claim that $V(p, q, y) = f(y) + g(p, q)$ and check whether this claim is true:

$$(18) \quad \left(\frac{y}{y^\circ}\right)^\eta f_y(y) = -\frac{g_p(p,q)}{x(p,q,y)}$$

Expression (18) is satisfied for any non-negative y and p holding q fixed. Note that the left-hand side term of (18) is a function of y and the right-hand side is a function of p. Therefore each side must be equal to a constant. Let both sides be equal to 1. Then from the left-hand side,

$$f_y(y) = (y^\circ)^\eta (y)^{-\eta} \Rightarrow f(y) = (y^\circ)^\eta \int_o^y \omega^{-\eta}\, d\omega.$$

Note that $f(y)_{y=0}$ is equal to zero due to zero utility. And from the right-hand side,

$$g_p(p,q) = -x(p,q,y^\circ) \Rightarrow g(p,q) = \int_p^\infty x(\tau,q,y^\circ)d\tau.$$

$$(4\text{-}5) \quad V(p,q,y) = f(y) + g(p,q).$$

The income elasticity of the private good demand, η, is

$$(4\text{-}6) \quad \eta = \frac{\partial x}{\partial y}\frac{y}{x} = -\frac{\partial[g_q(p,q)/f_y(y)]}{\partial y}\frac{y}{x} = [-\frac{g_p(p,q)}{f_y(y)}]\,[-\frac{f_{yy}(y)}{f_y(y)}]$$

$$= -\frac{f_{yy}(y)}{f_y(y)}\,y.$$

Comparison of expression (4-4) which (4-6) shows that they are equivalent except for the first term of the right-hand side of expression (4-4). This term vanishes in expression (4-6) due to the form of indirect utility function specified in expression (4-5). Therefore the property of the income elasticity of private good demand depends on the functional form of f(y). For example, homothetic preferences give

$$[(f_y(y)]^{-1} = y \Rightarrow f_y(y) = \frac{1}{y} \Rightarrow f(y) = \ln y, \quad \eta = 1.$$

satisfies restriction (8). And restrictions (8) and (9) together imply that the partial derivative of the private good demand with respect to the level of the public good is equal to zero at $\bar{\bar{p}}$, i.e., $x_q = 0$ at $p = \bar{\bar{p}}$. Therefore the following sets of restrictions are equivalent:

$$\left.\begin{array}{l} \partial(V_q/V_p)/\partial y = 0 \\ \partial(V_q/V_y)/\partial p = 0 \quad \text{at } p = \bar{\bar{p}} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \eta \text{ is constant} \\ x_q = 0 \quad \text{at } p = \bar{\bar{p}}. \end{array}\right.$$

Therefore what we need for analysis are a private good demand function with constant-income-elasticity and the value of $\bar{\bar{p}}$ at which $x_q = 0$. The key point once again is that for the analysis we do not need a specific utility function which is unobservable but a private good demand function which is estimable.[6]

The following considerations are important before this approach can be applied. First, finite threshold prices with two individual behaviors ($\bar{p}$, $\bar{\bar{p}}$) can be consistent with the properties of a well-behaved indirect utility function once the implicit assumption that the upper limit of the relevant price range with each individual behavior is a threshold price itself is adopted. Second, if the private good demand function is hyperbolic, then two threshold prices will be infinite. In this case we don't need an individual behavioral assumption, because the upper limit of the price will be infinite. Third, if the quantity of the private good demanded is zero at a finite price, then that price will be upper limit in calculating the amount of willingness-to-pay.

[6] There is an interesting paper by Morey (1984) concerning this issue. According to him, there are two approaches to estimating demand equations. The one approach specifies an algebraic form for the utility function, derives the corresponding system of demand equations, and only then uses data to estimate the coefficients in those demand equations. Imposing an algebraic form on the utility function with complete ignorance of data amounts unfortunately to imposing a priori restrictions on the preference ordering. These a priori restrictions are embedded in the demand equations. Since their imposition was not based on the data, the estimated demand equations are not necessarily correct and the resulting ranking could be wrong. The other approach to demand estimation specifies the algebraic form for the demand equations directly and then uses the market data to estimate their coefficients. The algebraic form of the demand equations determine he form of the utility function up to a monotonic transformation once the demand equations satisfy the integrability condition (Hurwicz and Uzawa, 1971)-negative semidefinitness of the substitution term matrices.

References

Bradford, D. and G. Hildebrandt, "Observable Preferences for Public Goods," *Journal of Public Economics,* 1977, 111-131.

Deacon, R., "Private Choice and Collective Outcomes: Evidence from Public Sector Demand Analysis," *National Tax Journal,* 30, 1977, 371-386.

Deaton, A. and J. Muellbauer, *Economics and Consumer Behavior,* Cambridge University Press, Cambridge, 1980.

Hanushek, E., "The Demand for Local Public Services: An Explanatory Analysis," *Public Needs and Private Behavior in Metropolitan Areas,* Jackson, J.E., ed., Balinger Publishing Company, 1975, 121-137.

Hildebrandt, G. and T. Tregarthen, "Observing Preference for Educational Quality: The Weak Complementarity Approach," *Scandinavian Journal of Economics,* 81, 1979, 168-197.

Hurwicz, L. and H. Uzawa, "On the Integrability of Demand Functions," *Preferences, Utility and Demand,* Chipman, J.S., et al., eds., Harcourt Brace Jovanovich, 1971, 114-148.

Morey, E., "Confuser Surplus," *American Economic Review,* 74, 1984, 163-173.

Oates, W., "The Effects of Property Taxes and Local Public Spending on Property Values: An Empirical Study of Tax Capitalization and the Tiebout Hypothesis," *Journal of Political Economy,* 77, 1969, 957-971.

Varian, H., *Microeconomic Analysis,* W. W. Norton & Company, New York, 1978.

Willing, R., "Incremental Consumer's Surplus and Hedonic Price Adjustment," *Journal of Economic Theory,* 17, 1978, 227-253.