Optimal Control of Economic Systems - Formulation of the Problems -

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1. Introduction

In recent years, there has been a growing interest in optimal control theory, in particular Pontryagin's maximum principle[1] and Bellman's dynamic programming,[2] as a possible planning tool for economic stabilization. Given an econometric model that is a reasonable description of the behavior and structure of the economy so that the model can be used for short-run forecasting purposes; and given a performance index or an objective functional[3] that is designed to represent the goals of economic stabilization such as growth without fluctuation in national income, full-employment without price inflation and so on; then the design of a stabilization policy can easily be thought of finding the solution of an optimal control problem of a system.

Essentially, the formulation of an optimal control problem requires: 1) a mathematical description of a system in terms of differential (or difference) equations into which stabilizers are introduced; 2) a statement of physical constraints on both control and controlled variables; 3) a set of boundary conditions on the variables in the model; and 4) specification of a performance criterion or an objective functional which is to be optimized.[4]

In applying the control theory algorithm to a real problem of economic planning, the system equations can be represented by an econometric model of the economy. The boundary conditions are the initial values of variables, and the desired values at the terminal time of the planning span. The objective functional is a quantitative expression of the planner's goals. Finally, physically realizable con-

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1 L. S. Pontryagin, et al (1962).

2 R. E. Bellman (1952 ond 1957).

3 A functional is a real-valued function defined on a set of functions, that is, the domain is a set of functions. Intuitively, we might say that a functional is a "function of functions."

4 The term "control" and "controlled" variables are defined in the later section.

trols and controlled variables have their magnitude limitations. The purpose of this paper is to give a mathematical formulation of the problem of economic stabilization in terms of optimal control problem that is described loosely above.

Section 2 explains briefly what the control theory is concerned with and the conceptual relationships and analogies between the theory of automatic control and economic theoty. It also defines some control theory terminologies and their counterparts of economic theory. In Sections 3 through 5, system models, constraints on control and controlled variables, and performance measures which are needed in formulating the control problem are discussed, respectively.

2. Automatic Control and Economic Theory

The modern control theory can be viewed as the confluence of three diverse streems: the theory of servo-mechanisms, the calculus of variations, and the development of computers. The theory of servomechanisms, which is now known as "classical control," is in general a trial-and-error process in which various methods of analysis are used iteratively. The theory of servomechanisms has found its most refined application in the design of electronic computers. It has been discovered that there exists profound analogies among the functioning of servomechanisms, electronic computers and the functioning of living organisms.[5] The principles of the functioning of self-regulation in living organisms are the same as the principles of automatic regulations in technical equipment and machines. In addition to this, it has been shown that these instances of self-regulation may be presented by a common scheme and a common mathematical theory.[6] It has further been pointed out that the regulation and control of social and economic process can also be treated similarly.

Indeed, the free-enterprise economy has long been regarded in economic theory as an automatically regulated system or as a self-regulating system through the mechanism of price and wage adjustment. The principles of government regulation and control of an economy can also be presented by the use of the same theoretical scheme which applies to automatic regulation and control in techniques.

Before proceeding further, it would be good to define a few words that appear in control theory literature. The human operator, in techniques, has control over certain variables; and those variables under human control are called control variables or inputs. This study will refer to these variables as stabilizers: examples in economic systems are government expenditures, taxes, money supply

[5] T. Milsum (1966).

[6] of earliest writers on the argument is Norbert Wiener (1948).

and so on.[7] The physical variables which are desired to be controlled or regulated are referred to as controlled variables or outputs. This paper will use the same term for the controlled variables: examples in economic systems are the level of national income, the rate of price change, the unemployment rate and the rest.

The basic concept in the science of regulation or control is the "feed-back" which can be defined as that property of a closed-loop system[8] which permits the output or some other controlled variables of the system to be compared with the reference or desired level of the variables so that the appropriate control action may be formed as some function of the output or the reference level. More generally, "feed-back" is said to exist in a system in which a closed sequence of cause-and-effect relationships exists between system variables. Providing an example would help to grasp what the "feed-back" relationships imply in economic terms.

The feedback relationship can be seen in a multiplier-accelerator system. Consider the familiar multiplier-accelerator model of Harrod-Domar type in continuous terms:

$$Y_t = C_t + I_t + A \tag{2-1}$$

$$C_t = cY_t \tag{2-2}$$

$$I_t = v(dY_t/dt) \tag{2-3}$$

giving

$$-vDY_t + (1-c)Y_t = A \tag{2-4}$$

where

Y_t : national income;
C_t : consumption expenditures;
I_t : investment expenditures;
A : autonomous expenditures;
c : marginal propensity to consume;
v : accelerator coefficient; and
D : d/dt.

In Equations (2-1) through (2-3), the time variable t is introduced,

7 In economic literature, the term "automatic or built-in" stabilizers usually refers to the fiscal instruments and these stabilizers are the institutional results of an economy's structure. For monetary variables, a fixed and auomatic annual increase in the stock of money has long been proposed, and endorsed by many of monetary economists, especially by Milton Friedman (1960) and by Edward S. Shaw (1958).

8 A closed-loop control system is one in which the control action is somehow dependent on the output.

thus giving a dynamic version of an economic system. The schematic form of the system is shown in Figure 2-1, which is commonly referred to as the block-diagram[9] of a system.

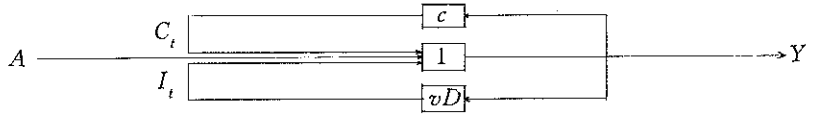

Figure 2-1

Figure 2-1 shows that: the level of national income Y_t is influenced by the consumption C_t and the investment I_t in which C_t and I_t are, in turn, influenced by Y_t; that is, there exist "feed-back" relationships between income Y_t, and consumption C_t and investment I_t, respectively. These directions of influence are indicated in Figure 2-1 by arrows.

Equation (2-4) is a differential equation, or a final equation in control theory terminology, the solution of this equation giving the time path of changes in national income Y_t as time t varies. The solution of Equation (2-4) is given by:

$$Y_t = Y_0 e^{\left(\frac{1-c}{v}\right)t} + A\left\{\frac{1}{1-c}\left[1 - e^{\left(\frac{1-c}{v}\right)t}\right]\right\} \qquad (2\text{-}5)$$

where Y_0 is the initial value of Y_t when t=0. Equation (2-5) describes the dynamic relationship between the level of national income Y_t and autonomous expenditures A. This relation is depicted in Figure 2-2.

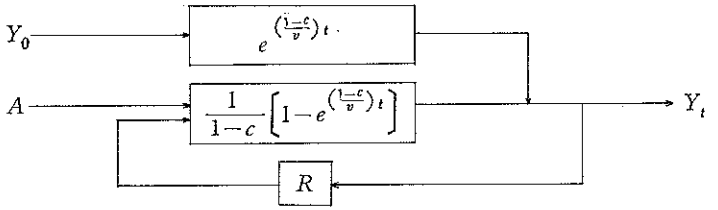

Figure 2-2

Two boxes, $e^{\left(\frac{1-c}{v}\right)t}$ and $\frac{1}{1-c}\left[1-e^{\left(\frac{1-c}{v}\right)t}\right]$, in Figure 2-2 represent the whole system of the multiplier-accelerator model.

The problem of controlling or regulating the level of national

9 Block-diagrams are shorthand, graphical representation of a physical system or the set of mathematical equations characterizing its parts.

income Y_t to maintain it in a desired level in an automatic fashion is the problem of designing a stabilizer R. In a practical application of the theory of automatic control, it is usual to assume that the boxes in Figure 2-2 are given and determined by the existing system, and the box R is constructed appropriately by man, coupled in some way with the regulated system. Indeed, the feature of the time path of national income Y_t in Equation (2-5) depends on the values of the marginal propensity to consume c and the accelerator coefficient v, and on the level of autonomous expenditures A; and those constants cited above characterize the behavior of each sectors of a model economy of Harrod-Domar. The question is what type of the stabilizer R to be, and how to obtain the functional form or the time path of R.

As mentioned earlier, the theory of servomechanisms is in general a trial-and-error process. The first theoretical tools used were based upon the work done by Bode and Nyquist. Concepts such as frequency responses, band width, gain, and phase margin were used to design servomechanisms in the frequency domain. Much of the early work in control theory was concerned with one-input and one-output relationship. The classical literature on control theory is largely concerned with the search for criteria with which to judge the usefulness of the control system of one-input and one-output.[10] Radically different performance criteria must be satisfied, however, by a complicated, multiple-input and multiple-output system.

The need for the new approach has led the early control theory to the modern control theory through the use of the calculus of variations: in particular, its extension in the form of Pontryagin's "maximum principle"[11] and Bellman's "dynamic programming."[12] The main aims of modern control are to deempiricize control design problems and to present solutions to a much wider class of control problems such as multiple-input and multiple-output systems with con straints on input and output variables. These aims are achieved by the use of mathematical discipline of the calculus of variations.

The calculus of variations is a field of mathematics which is concerned with the finding of a function that optimizes a given functional. The modern control techniques such as the maximum principle of Pontryagin and the dynamic programming of Bellman are derived from significantly different point of views, but it is now becoming clear that they are inspired to a great extent by the classical calculus of variations.

10 These criteria are collected in S. M. Shinners (1964).

11 L. S. Pontryagin (1962).

12 R. E. Bellman (1952 and 1957).

> As Michael D. Intriligator put it,[13]
> "The basic problem of economics, economizing, is that of allocating scarce resources among competing ends. . . . . .
> . . . . . .
> The economizing problem can be considered the application to economics of the mathematical optimization problem, defined as the choice of values of certain variables so as to maximize a function subject to constraints,"

the policy problem for an economy as a whole can be viewed as the problem of optimizing a "social welfare function" which depends on the level of economic variables such as production, employment, price level, and economic growth, subject to constraints of an economic system behavior. If the system equations are represented in dynamic terms as are the usual econometric models, the problem of choosing an appropriate policy mix is that of optimal control.

While it is beyond the planned scope of this study to discuss the capability of the computers in control theory, it is clear that they are now the most important tool. There is no doubt that the control design is benefited by the general availability of computers. Indeed, in many cases, particularly in a large-scale system seeking numerical solutions, a computer is an integral part of the modern control problems.

3. Economic System Equations

The mathematician Laplace is reputed to have said, "Give me only equations of motion, and I will show you the future of the universe." Likewise, in order to evaluate the problem of stabilization policy with the use of control theory discipline, it is first necessary to ascertain the behavioral relationships and the structure of an economy into which such policies are introduced. This is so because the motion of an economic system responding to policy actions should adequately be described in the model or equations of the relationships and structure. Since there are innumerable number of variables to be considered in an economic system, the usual practice is to describe a system by a simplified model: a simplified model in terms of the number of equations and variables in the system model.

It is further assumed that:

1) The policy authorities have an econometric model that is a reasonable description of the behavior and structure of the economy into which policies are applied. That is, the model reflects fairly well the dynamic and lagged behavior of the economy so that the model can be used for short-run forecasting (one to three years) purposes;

13 Michael D. Intriligator (1971).

2) The model is linear or linearized, and time-invariant.[14] Linearity in economic relationships gives economists a simplified form of the equations that facilitate analysis. Linearity is, however, only for a convenience and must be sacrificed in favor of reality. To estimate statistically the unknown parameters on the basis of observed data, however ,we usually assume linearity at least in the parameters. Non-linearity in variables can of course be easily handled statistically, and the technique for linearization of such non-linear terms is available in mathematical as well as in economic literature.[15] Obtaining the final equations of a non-linear model, however, poses a difficult mathematical problem; and

3) The mode lis deterministic. In fact, most equations in an econometric model are not really deterministic. The only exceptions are definitions or identities in the model. The estimated coefficients based on observed data are themselves random variables and estimated equations in the model have implicit error terms. Although any stochastic system can be treated for control problems, it has been well known that analytical and computational difficulties involved in a stochastic system would make its applications to an econometric model impractical. The issue must, instead, be how well we could specify a model and estimate parameters in the model, rather than the issue whether the model be stochastic or deterministic.[16]

State-Form Representation of a System

As already noted, any system containing lagged variables has a dynamic character, implying that the level of endogenous variables in a model is an implicit function of the time variable t. To examine the dynamic response of a control on a particular endogenous variable, it is necessary to eliminate algebraically all endogenous variables, both current and lagged, other than the one interested. The resulting equation so obtained has been called the final equation.[17] The final equation is actually a difference (or a differential) equation in one endogenous variables, expressing the current value of the variable as a function of its lagged values and control variables. For a difference (or a differential) system, when for any reason analysis in the time domain is preferred as is the case in most economic time series, the use of state-space approach offers great conveniences for control problem. The complete examination of the state-space approach is beyond of the scope of this study. Providing the definition of the state variable and methods of obtaining the state-form equations

14 A time-invariant differential (or difference) equation is a differential (or difference) equation in which none of the terms depends explicitly on the independent variable time t.

15 For the linearization of a non-linear model, see Arthur S. Goldberger (1959).

16 The reasoning for a completely deterministic model is well explained in Robert S. Pindyck (1972).

17 The term "final equation" was first named by J. Tinbergen (1939).

would, however, be helpful to grasp the control problem.[18]

The state of a system is a set of quantities, say, $x_{1,t}, x_{2,t}, \cdots\cdots\cdots\cdots\cdots x_{n,t}$, which if known at time t = t_o are determined for all time $t > t_o$ by specifying the inputs to the system for all time $t > t_o$. Let the inputs, or stabilizers, $u_{1,t}, u_{2,t}, \cdots\cdots\cdots, u_{m,t}$ be represented by an input vector

$$U_t = (u_{1,t}\ u_{2,t} \cdots\cdots u_{m,t})^T \tag{3-1}$$

and the outputs $y_{1,t}, y_{2,t}, \cdots\cdots\cdots\cdots, y_{r,y}$ by an output vector

$$Y_t = (y_{1,t}\ y_{2,t} \cdots\cdots y_{r,t})^T \tag{3-2}$$

The input and output vectors are assumed to be the functions of the time variable t as implied by the notation of U_t and Y_t.[19] In symbols, the definition of the state of a system implies that the state vector X_t can be written by:

$$X_t = (x_{1,t}\ x_{2,t} \cdots\cdots x_{n,t})^T \tag{3-3}$$

$$= F(X_{t_o};\ U_t) \tag{3-4}$$

where F is a single-valued function of its arguments X_t and U_t. The output vector is then written by:

$$Y_t = G(X_{t_o}; U_t) \tag{3-5}$$

where G is again a single-valued function of its arguments. Equations (3-4) and (3-5) constitute the state-form of a system.

Turn to, now, the methods of obtaining the state-form of a system.[20] Consider for simplicity a time-invariant linear difference equation with one-input and one-output:

$$\begin{aligned} &a_n y_{t+n} + a_{n-1} y_{t+(n-1)} + \cdots\cdots + a_1 y_t \\ &= b_n u_{t+n} + b_{n-1} u_{t+(n-1)} + \cdots\cdots + b_1 u_t \end{aligned} \tag{3-6}$$

where

18 This is so because of a number of reasons: 1) a difference (or a differential) equation in the form of state variable is ideally suited for digital or analog solution; 2) the state-form provides a unified framework for the study of non-linear as well as linear systems; and 3) the state-form is invaluable in theoretical investigations.

19 If system equations are continuous, the input and output vectors are also continuous in time t. Likewise, the vectors are discrete in time t if the system equations are discrete.

20 There are a number of different methods of obtaining the state-form equations of a system. See LaMar K. Timothy (1968); E. Polak (1966); and Donald M. Wiberg (1971).

Y_t : an output variable;
U_t : an input variable;
a,b : constant coefficients; and
n : the number of periods lagged.

Letting E denote the difference operator where $Y_t = E y_{t-1}$, and rearranging Equation (3-6), it is obtained that:

$$y_t = \frac{b_n}{a_n} u_t + \frac{1}{E}\left[\frac{b_{n-1}}{a_n} u_t - \frac{a_{n-1}}{a_n} y_t\right] + \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots + \frac{1}{E^n}\left[\frac{b_1}{a_n} u_t - \frac{a_1}{a_n} y_t\right] \qquad (3\text{-}7)$$

The flow diagram of Equation (3-6) can be drawn from Equation (3-7) starting with the output Y_t at the right and working to the left, and the diagram is depicted in Figure 3-1.

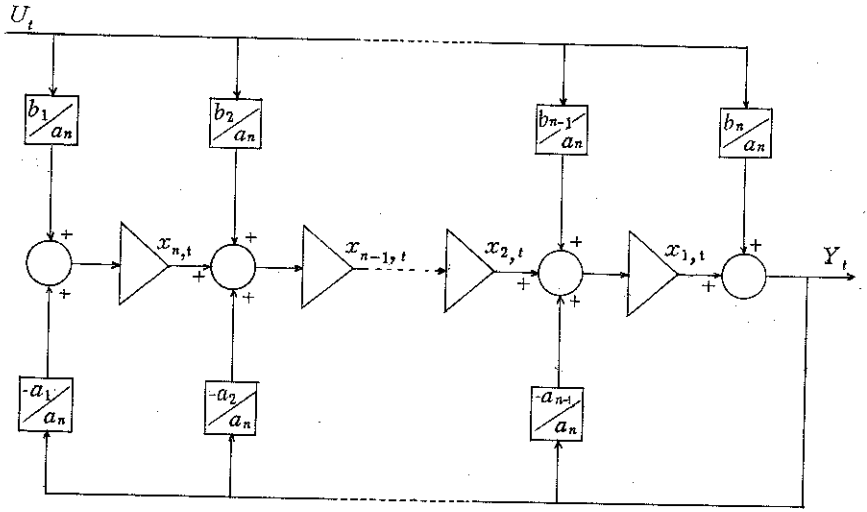

Figure 3-1

○ : summer
□ : scalar
▷ : delayer

The output for each delayer is labeled as state variables

$x_{1,t}, x_{2,t}, \ldots\ldots\ldots, x_{n,t}$. The summer equations for the state variables have the form of:

$$y_t = x_{1,t} + \frac{b_n}{a_n} u_t \tag{3-8}$$

and

$$x_{1,t+1} = -\frac{a_{n-1}}{a_n} y_t + x_{2,t} + \frac{b_{n-1}}{a_n} u_t$$

$$x_{2,t+1} = -\frac{a_{n-2}}{a_n} y_t + x_{3,t} + \frac{b_{n-2}}{a_n} u_t$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3-9}$$

$$x_{n,t+1} = -\frac{a_1}{a_n} y_t + \frac{b_1}{a_n} u_t$$

By eliminating Y_t in Equation (3-9) by Equation (3-8), it is obtained that:

$$x_{1,t+1} = -\frac{a_{n-1}}{a_n} x_{1,t} + x_{2,t} + \left(-\frac{a_{n-1} b_n}{a^2_n} + \frac{b_{n-1}}{a_n}\right) u_t$$

$$x_{2,t+1} = -\frac{a_{n-2}}{a_n} x_{1,t} + x_{3,t} + \left(-\frac{a_{n-2} b_n}{a^2_n} + \frac{b_{n-2}}{a_n}\right) u_t$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{n,t+1} = -\frac{a_1}{a_n} x_{1,t} + \left(-\frac{a_1 b_1}{a^2_n} + \frac{b_1}{a_n}\right) u_t$$

(3-10)

In a matrix form, Equations (3-8) and (3-10) can be written by:

$$X_{t+1}=AX_t+BU_t \tag{3-11}$$

$$Y_t=CX_t+DU_t \tag{3-12}$$

where

$$X_t=(x_{1,t}\,x_{2,t}\cdots\cdots\cdots\cdots x_{n,t})^T;$$

$$U_t=u_t;$$

$$Y_t=y_t;$$

$$A=\begin{pmatrix} -\frac{a_{n-1}}{a_n} & 1 & 0\cdots\cdots\cdots 0 & 0 \\ -\frac{a_{n-2}}{a_n} & 0 & 1\cdots\cdots\cdots 0 & 0 \\ \cdots\cdots & & & \\ \cdots\cdots & & & \\ -\frac{a_1}{a_n} & 0 & 0\cdots\cdots\cdots 1 & 0 \end{pmatrix};$$

$$B=\begin{pmatrix} -\frac{a_{n-1}b_n}{a^2{}_n}+\frac{b_{n-1}}{a_n} \\ -\frac{a_{n-2}b_n}{a^2{}_n}+\frac{b_{n-2}}{a_n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ -\frac{a_1b_n}{a^2{}_n}+\frac{b_1}{a_n} \end{pmatrix};$$

$$C=(1\quad 0\quad 0\cdots\cdots\cdots 0)\ ; \qquad \text{and}$$

$$D=\frac{b_n}{a_n}$$

Equations (3-11) and (3-12) constitute the state-form of the system of Equation (3-6). The constant matrices A, B, C, and D are determined by the coefficients of the system equations, a's and b's. By using the same procedures discussed above or other methods, any system model can be reduced to the state-form equations which take the same form of Equations (3-11) and (3-12).

As noted previously, not all of endogenous variables in a model appear in the state vector X_t, but only those variables that to be

controled. Suppose an econometric model contains three endogenous variables, namely national income, consumption and investment as was the case in the Harrod-Domar type of multiplier-accelerator model, with two behavioral equations corresponding to consumption and investment, and an income identity. Suppose that it is desired to maintain the level of income in a certain rate of growth. Consumption and investment, then, are eliminated to obtain the final equation, a difference (or a differential) equation in terms of national income. The state vector X_t so obtained contains only the variable of national income.

4. Constraints on Control and Controlled Variables

After obtaining an appropriate model of an economy into which a mix of policy instruments to be introduced, the next step to be done in formulating control problems is to define the constraints on both control and controlled variables in the system model. In any realistic economic system, such constraints commonly occur.

The problem of assigning constraints on controlled variables arises because of the structural limitation of an economy and of the very nature of the controlled variables themselves: the rate of growth in output of any economy has its own limits; the rate of unemployment could not be less than a certain percentage, say, four(4) percent at its minimum. More importantly, the range of policy objectives are not unlimited: the rate of growth in national income is desired to exceed the rate of population increase so that per capita income to increase; the rate of unemployment must remain around four(4) percent; the rate of price change is desired to keep around a certain percent, say, three(3) percent annually.

Physically realizable controls also have their magnitude limitations: changes in government expenditures cannot be undertaking without institutional constraints; the level of tax revenue collected depends on inflexible tax structures; and money supply must not be subject to unlimited changes in one time and another. One reason is the fact that policy making is divided between a number of different authorities. Fiscal policy is divided, for example, between Congress and the Administration: tax rates and defiicit ceilings are set by congress and the amount of money for various government agencies are appropriated by Congress, too. Monetary policy is also divided between Congress and the Federal Reserve system in the case of the U.S.: while the Federal Reserve System executes monetary policy, it is ultimately responsible to Congress.

The question, then, is how to impose the limitations on control and controlled variables. The long history of past experiences could tell us what values of those limitations be; and theory has suggested many different ideas on the subject, from which a range

of constraints on policy instruments could be set. Considering these factors together, policy authorities could make a reasonable assumption on the realistic values of each control and controlled variables, their upper and lower limits. These limits are expressed in symbols by:

$$\bar{X}_t^0 \leqq X_t \leqq \bar{X}_t^1 \tag{4-1}$$

$$\bar{U}_t^0 \leqq U_t \leqq \bar{U}_t^1 \tag{4-2}$$

where

X_t : the state vector;
$\bar{X}_t^0$: the lower limit of state vector; X_t ;
$\bar{X}_t^1$: the upper limit of state vector; X_t ;
U_t : the control vector;
$\bar{U}^0$: the lower limit of control vector; U_t ; and
$\bar{U}_t^1$: the upper limit of control vector. U_t .

5. Performance Indices[21]

The last step in formulating a control problem is to select an appropriate performance measure. That is, the designer of stabilizers for a system must find a mathematical expression to measure how the system be optimal in comparison with other systems. For the problem of designing optimal economic policy, the expression should be in terms of economic goals to be achieved; the goals such as growth without fluctuations in national income, and full-employment without price inflation, but both with minimum policy efforts to achieve the goals. A good performance index is, however, difficult to define.

Ideally, a system would be desired to carry out its command without error. For real world problems, however, this is almost not achievable. What can be hoped for is to design a control to perform as closely as possible to the ideal. In practice, the designer of stabilizers sets up a mathematical performance measure to find solutions that optimize the particular measure. In this study, the problem of economic stabilization is considered as a tracking and minimum-control-effort problem in control theory literature.

The Tracking Problem

21 The terms, performance "measures", "indices" or "criteria" are concurrently used with having the same meaning in control theory literature.

The tracking problem in control is to maintain a state vector X_t as close as possible to the desired state $\hat{X}_t$ during a given time interval $[O, N]$, where N is the number of planning periods. Suppose, for example, that the level of national income y_t is desired to track a desired level $\hat{y}_t$ which may be defined as the level with a certain rate of growth.[22] Then, one can select as a performance index:

$$J_1 = h(y_N - \hat{y}_N)^2 + \sum_{t=0}^{N-1} q(y_t - \hat{y}_t)^2 \qquad (5\text{-}1)$$

where h and q are weighting constants. It can be seen that J_1 represents the sum of the square of deviations between the desired level of income $\hat{y}_t$ and the actual income y_t. The weighting constants, h and q, could be the same values, but if the terminal stage of the planning periods is considered to be more important than the intermediate stages in terms of the closeness to $\hat{y}_t$, the constant h must be a greater value than the constant q.

For a multiple-input and multiple-output system, the performance index J_1 for a tracking problem can be expressed in a similar way by using the input and output vectors:

$$J_1 = (X_N - \hat{X}_N)^T H(X_N - \hat{X}_N) + \sum_{t=0}^{N-1} (X_t - \hat{X}_t)^T Q(X_t - \hat{X}_t) \qquad (5\text{-}2)$$

where

X_t : the actual level of the state vector;
$\hat{X}_t$: the desired level of the state vector X_t;
Q, H: weighting matrices;
N : the number of planning periods;
t : the time variable; and
T : the transpose of a matrix.

The elements of the weighting matrices Q and H represent the relative contributions of deviations between actual and desired levels of the state vectors to the performance index J_1.[23]

22 $\hat{y}_t$ might be in the form of $y_0(1+r)^t$, where y_0 is the level of national income at the beginning period of planning, and r is the rate of growth in national income, say, five(5) percent per year.

23 In connection with the tracking problem, another important class of control problem is the regulator problem. This is the problem to apply a control or controls to take a system from a non-zero state to zero state. Suppose, for example, that the rate of price

The Minimum-Control-Effort Problem

The problem of minimum-control-effort is to transfer a system from an arbitrary state to a specified target with minimum expenditures of control effort. In economic terms, this is the problem of minimizing the costs involved in the policy actions to achieve the goals of economic stabiliaztion. In fact, any policy actions are not free in terms of costs involved; there are involved in policy actions, for example, administrative costs, and these costs are increasing because of the presence of bureaucracy. The less the level of government expenditures is, the less the costs involved in its administration are; and thus the better off the society as a whole would be, if the same goal could be achieved with the less of the government spending.

More importantly, there are another costs involved to avoid; the costs of excess burden. The costs of public services provided by government agencies must be attributed to the members of the society in accordance with the society's individual preference patterns. The tax-expenditure plan should be made to accomplish certain objectives to maximize the whole society's welfare; but it should not interfere with the functioning of the free market system in which individuals act to maximize their own welfare.

Keeping the materials discussed above in mind, one may select a performance index for the minimum-control-effort in the form of:

$$J_2 = \sum_{t=0}^{N-1} U_t^T R U_t \qquad (5\text{-}3)^{24}$$

where

U_t : the control vector;

R : a weighting matrix;

N : the number of planning periods; and

T : the transpose of a matrix.

Each elements of the weighting matrix R give the relative importance of each control to the performance measure J_2.

Since the concern in the problem of economic stabilization is the growth and stability, together with minimum policy effort to achieve

increase is presently non-zero, say, four(4) percent. Suppose then that a policy maker wishes the rate of price change to be maintained at zero percent. It can be seen, however, that this problem is exactly the same as the tracking problem if one element of the desired state vector corresponding to the price variable is set zero. In fact, the regulator problem is a special case of the tracking problem.

24 It should be pointed out that Equation (5-3) is a significantly different form of the index from the usual viewpoint appeared in economic literature. See Gregory C. Chow (1972) and Robert S. Pindyck (1972).

the goal of growth and stability and to correct fluctuations in economic activity, the weighted sum of two indices J_1 and J_2, each corresponding to the tracking and minimum-control-effort problem, would be appropriate as a performance measure for the problem of economic stabilization. The measure can therefore be expressed in the form of:

$$J=J_1+J_2$$

$$=(X_N-\hat{X}_N)^T H(X_N-\hat{X}_N)+$$

$$\sum_{t=0}^{N-1}\left\{(X_t-\hat{X}_t)^T Q(X_t-\hat{X}_t)+U_t^T RU_t\right\} \qquad (5\text{-}4)$$

Equation (5-4) is the cost functional to be minimized in designing optimal economic stabilizers.

6. The Mathematical Statement of The Problem

With the background materials discussed in mind, it is now possible to present the precise mathematical statement of the problem of planning optimal economic policies:

"Minimize : $$J=(X_N-\hat{X}_N)^T H(X_N-\hat{X}_N)+$$

$$\sum_{t=0}^{n-1}\left\{(X_t-\hat{X}_t)^T Q(X_t-\hat{X}_t)+U_t^T RU_t\right\} \qquad (6\text{-}1)$$

Subject to : $$X_{t+1}=AX_t+BU_t \qquad (6\text{-}2)$$

$$\overline{X}_t^0 \leqq X_t \leqq \overline{X}_t^1 \qquad (6\text{-}3)$$

$$\overline{U}_t^0 \leqq U_t \leqq \overline{U}_t^1 \qquad (6\text{-}4)$$

X_0 given."

The problem is to find a control law or the time path of economic stabilizers U_t^* (t = 0,1,2, ..., N-1) which causes the economic system of

$$X_{t+1}=AX_t+BU_t \qquad (6\text{-}5)$$

to minimize the performance index J of Equation (6-1).

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